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LETTER
TO
HIS GRACE THE DUKE OF BUCCLEUCH,
PRESIDENT ELECT
OF THE
*BRITISH ASSOCIATION FOR THE
ADVANCEMENT OF SCIENCE,*
1867—1868,
ON THE
QUADRATURE AND RECTIFICATION
OF THE CIRCLE.

By JAMES SMITH, Esq.,

CHAIRMAN OF THE LIVERPOOL LOCAL MARINE BOARD, AND MEMBER OF THE
MERSEY DOCKS AND HARBOUR BOARD.

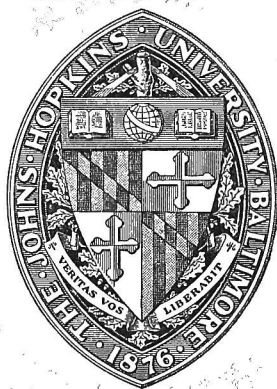
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1867.

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THE
QUADRATURE AND RECTIFICATION
OF THE CIRCLE.

BARKELEY HOUSE, SEAFORTH,

29th July, 1867.

To His Grace the DUKE OF BUCCLEUGH.

My Lord,

May it please your Grace,

On the 17th ult., I made known to your Grace, that I was about to address a letter to you as the President elect of "*The British Association for the Advancement of Science*,"* on the *Quadrature and Rectification of the Circle*." It was my intention to throw off the letter in the form of a pamphlet, for the purpose of distribution among the assembled Members and Associates at the forthcoming meeting of the Association; and at the time of writing to your Grace, I thought I should be able to place a copy of it in your hands before the middle of August. With this in view, I had so far proceeded with my design

* See Appendix.

as to have one sheet printed off, and another in type, when I found that if I followed out the plan upon which I had commenced, it would cease to be a letter, or even a pamphlet, and become a volume of no mean dimensions. Now, however strong may be my conviction that—as *an old Life Member of the British Association*—I am justified in addressing your Grace on questions strictly within, and carefully confined to, the professed objects of that Association; I am at the same time conscious, that I have no right to trespass upon your Grace's valuable time, beyond such reasonable limits as fairly come within the province of the President for the time being. This forced upon me the necessity of reconsidering my design, and after much thought and reflection it occurred to me, that I might accomplish all I think essentially necessary in the cause of Scientific truth within the limits of a letter; and thus, avoid trespassing unreasonably upon the numerous occupations of your Grace.

I shall endeavour so to frame this communication that I may make it an introductory chapter to a larger work—of which, what is already in print might form a part—if at some future time I should make up my mind to complete and publish it, out of the abundant materials I have in my possession. This, however, is problematical. Such a work would no doubt serve to shew, that the Geometers and Mathematicians of the nineteenth century were a “*confraternity of men*” with heads so full of prejudice engendered by “*crammed erudition*,” that there was not left “*a cranny hole for reasoning to get in at* :” * and it would be, not only a literary curiosity, but to Mathematical students of another generation, of considerable service.

* See *Athenæum*, May 11, 1861. Article—The Quadrature of the Circle: Correspondence between an eminent Mathematician and James Smith, Esq.

This, however, I reserve for further consideration. If I should be induced to complete the work, it will not be with a view to profit—for publishing on this subject I have found very unprofitable as a commercial speculation—but from a sense of duty.

I may inform your Grace that the only *plausible argument* I have ever heard advanced against the truth of the “*theory*,” that 8 circumferences of a circle are exactly equal to 25 diameters, is distinctly and fairly stated in the three following letters, which are taken from a London periodical, *The Correspondent*, which I regret to say is now extinct :—

QUADRATURE OF THE CIRCLE.

SIR,—As Mr. Smith wishes the public to accept the fact which he believes he has proved—viz., that the true value of $\pi = \frac{25}{8}$, may I be permitted to ask him a question which seems to bear very closely on the subject?

Supposing the diameter of a circle to be 1 foot, what is the perimeter of an inscribed regular polygon having 18 sides?

In my attempts to solve this question I have arrived at a result which seems worthy of notice. Now, the side of the polygon subtends an angle of 20° , and if we denote the side by a , we have at once (the radius being r)—

$$\frac{1}{2} a = r \sin. 10^\circ,$$

$$\text{Or, since } r = \frac{1}{2},$$

$$\frac{1}{2} a = \sin. 10^\circ.$$

The value of this sine is given in “Hutton’s Logarithmic Tables” as = .1736482. Multiply by 18, and we get the perimeter of the figure = 3.1256676 feet.

If this be correct, and Mr. Smith be also correct, it follows that the circumference of the circle (which he makes to be 3.125 feet) is *less* than the perimeter of the regular polygon which

it circumscribes. My only assumption is that of the value of $\sin. 10^\circ$. It is for Mr. Smith to say whether this value is incorrect or not, and, if incorrect, it is for him to set it right.

But my chief object is just to point out that we need not *theorize* about the matter at all. A plain practical man who does not understand mathematics, but who can just draw a diagram, may make a regular polygon of 18 sides for himself, and can tell by measurement that the perimeter, or whole way round it, is very nearly $= 3\frac{1}{8}$ of the diameter, or breadth across; if anything, the proportion is a little greater than $3\frac{1}{8}$. Here is a simple practical test for the general public to judge by—as I suppose it to stand to common sense that the circumference of a circle on the same diameter is *greater* than the perimeter of the 18-sided figure. But Mr. Smith would, apparently, make it *equal* or *less*.

Offering this test for the use of any one interested in the question,

I remain, Sir, yours very truly,

WALTER W. SKEAT.

A SLICE OF THE SEAFORTH MINCE PI.

SIR,—Mr. James Smith makes frequent use of Euclid's 47th proposition, which, I conclude, therefore, he admits, as well as the 20th. Moreover, his last letter but one informs us (*Correspondent*, p. 23) that no less than three natural sines, those of 30° , of 45° , and of 60° are "correctly given in our tables;" and to leave no doubt about their values, Mr. Smith himself states them as $\cdot 5$, as $\cdot 707107\dots$, and as $\cdot 866025\dots$

If, then, on any quadrant, whose radius may be called unity, we mark too points, at 30° and 45° from one extremity, join them, and draw the sine and co-sine to each; these latter four lines, he assures us, measure, the longest, $\cdot 866025$, &c., the two next, $\cdot 707107$, &c., and the shortest, $\cdot 5$.

Now the two points being also joined by a straight line, he

will allow this, I suppose, to be the chord of 15° , and also the slant side of a certain right-angled triangle, whose other sides are, by his showing,

$$\begin{array}{r}
 \text{The difference between } \cdot 866025\dots \\
 \text{and } \cdot 707107\dots \\
 \hline
 \text{namely } \cdot 158918\dots \\
 \text{and the difference between } \cdot 707107\dots \\
 \text{and } \cdot 5 \\
 \hline
 \text{namely } \cdot 207107\dots
 \end{array}$$

By squaring these two, then, $\cdot 158918\dots$ and $\cdot 207107\dots$, adding their squares together, and taking the root of the sum, we have (if by Mr. Smith's leave, the 47th proposition is not as great a delusion as the tables) $\cdot 261052\dots$ for the *chord* of 15° . But the *arc* of 15° is a twelfth of the arc of 180° , and may be had by dividing the Smithian π into twelve, thus:—

$$\begin{array}{r}
 12 \overline{) 3 \cdot 125000} \\
 \underline{\cdot 260417} = \text{arc of } 15^\circ \text{ (Smithian)} \\
 \text{which take from } \cdot 261052 = \text{chord of } 15^\circ \text{ (popular \& Smithian)} \\
 \hline
 \text{leaves } \cdot 000635 = \text{excess of chord over arc.}
 \end{array}$$

By $\cdot 000635$ of a radius, then, does this straight line exceed the curve (both Smithian) between the same two points! Not having read any earlier letter, I know not what are the “difficulties” out of which Mr. Smith calls on his opponents to help each other, but surely here is one that your readers have a right to see explained before hearing any more about “dogmatic blunderers,” &c. Is the non-“mysterious” $3 \cdot 125$ found a *mince* π after all (in the French sense of *mince*), and if so, how much is it to be eked out? Or is the arc of 15° *really* shorter than its chord?

I remain, Sir, yours very respectfully,

E. L. GARBETT.

THE QUADRATURE OF THE CIRCLE.

Sir,—I am the Cornish man referred to by Mr. Smith, and again by Mr. Garbett, in your last number. My suggestion was, that it would facilitate a settlement of the controversy if the following easy problem were thrown out for solution among your numerous readers, and it should (for our present purpose) be solved without using any tables of sines or their logarithms:—An isosceles triangle has each of its equal sides, unity (1). Its vertical angle is 15° . Find the base. This base will obviously be the chord of an arc of 15° , and $\frac{\pi}{12}$ will be the corresponding arc; we shall then be able to compare their magnitude. The work should be shewn at length. It may be assumed that $\sin 30^\circ = \frac{1}{2}$; from this, $\sin 15^\circ$ or co-sine 15° (and if employed in the calculation, $\sin 7^\circ 30'$) must be found. Our Tables of Sines are correct, but as this has been disputed, we save controversy by not employing them. Any one moderately versed in Trigonometry can solve the above problem, and its result on the value of π will be inevitable and conclusive.

I am, Sir, your obedient servant,

Launceston, Cornwall, 5th March, 1866.

G. B. G.

I was in correspondence with G. B. G.—arising out of a letter he addressed to me in January 1866, and who is personally unknown to me at this moment—at the time his letter appeared in the *Correspondent*. That gentleman, without solicitation on my part, had given me an assurance that he would not join the list of my public opponents in that Journal—of whom he said I had already enough—*without my sanction*, and I was greatly surprised

when I saw his letter for the first time in the columns of the *Correspondent*. After an exchange of upwards of 120 letters, our correspondence has just terminated. That your Grace may not think me chargeable with "*violating the courtesies of private life*," I give you a copy of his last letter :—

REV. GEORGE B. GIBBONS, B.A. to JAMES SMITH.

LANEAST, LAUNCESTON,

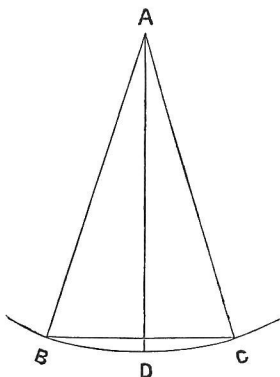
23rd May, 1867.

Dear Sir,

Though I did not expect, and do not wish, such a result of our correspondence, I am content to appear in print if you so decide it.

If $AC = AB = 1$, and $BAC = 15^\circ$, you have proved $BC = .261...$ You assert $\text{arc } BDC = 3\frac{125}{12}$

$= .2604$, and thereupon you use rather uncivil language to me, because I cannot consent to a value of π which would make the arc *shorter* than its chord? So be it: my betters have had worse treatment, and why should I be spared? Whewell, De Morgan, Airy, Hamilton, vastly my superiors in Mathematics, and known to fame—if they are treated as ignorant blunderers, why should I complain of anything you say concerning me?



I hope you will send me a copy of your book. With all good wishes for your health and happiness,

Believe me, dear Sir,

Yours very sincerely,

GEO. B. GIBBONS.

Your Grace cannot fail to observe the *insidious* care with which Mr. Gibbons has penned this letter. True! I did prove, and have over and over again admitted, that by making the sine of an angle of $30^\circ = \frac{1}{2} = \cdot 5$, a starting point; we may, by a certain application of the 47th proposition of Euclid's first book, apparently, make the line $BC = \cdot 261\dots$; but I have also proved in my letters to that gentleman, in a variety of ways, that there is a fallacy in this, and wherein that fallacy consists; and my last eight or ten letters were devoted principally to proofs by Logarithms. Will Mr. Gibbons dare to assert that he ever attempted to grapple with any one of my logarithmic proofs? I trow not! He knows that I have carefully preserved his letters, which speak for themselves, and can be forthcoming at any time. Can he say as much for mine? He *will* remember that, in the early part of our correspondence, my letters found their way to his waste-paper basket as soon as read, and he knows I have the proof. He *may not* remember, but I have the proof, that at a very late period of our correspondence he admitted that, up to that time, I had not written anything to him "*unbefitting a Christian and a gentleman.*" He now charges me with using "*rather uncivil language.*" This refers, and can only refer, to the following extract from my last letter to him, dated 20th May, 1867. "Your last letter, dated 11th inst. commences:—'*I have really exhausted all I have to say;*' and concludes by telling me what you have decided upon as to your future course, with reference to the interesting controversy in which we have been so long engaged, in the following words:—'*I shall receive willingly, and read (till I come to something I cannot understand) anything you favour me with. If I don't answer, it will be because I have already offered all I have to say.*' That letter is no

answer to mine of the 7th inst., but it furnishes a very distinct proof, that you neither understand the construction, nor the proper use of, Mathematical Tables, (*Mr. Gibbons had more than once told me that I did not understand the construction of these tables,*) or you would not have had the folly to say :—‘ I don’t see how A (*referring to a particular angle in a geometrical figure enclosed in my letter of the 7th May*) = $23^{\circ} 44'$.’ As I have no reply to my letters of the 9th and 11th inst., I may fairly infer, that ‘*you have already offered all you have to say,*’ on the *Quadrature of the Circle, the Solution of a Right-angled Triangle, and the Infallibility of Mathematical Tables*; and that so far as you are concerned, our correspondence is closed: and it now only remains for me to trouble you with a very few more letters, preparatory to publishing another volume.” These are the facts, upon which I must leave your Grace to form your own opinion; but I feel confident your Grace will agree with me in thinking, that it was necessary I should say so much in self-defence.

Well, then, your Grace will observe, that all these gentlemen are obliged to resort to, and make an application of, the 47th proposition of Euclid’s first book, in support of their *opinions*: or, if they be called arguments, they are arguments based on a *false assumption*, which no “*reasoning geometrical investigator*” can admit as a starting point in the solution of the problem: “*What is the ratio of diameter to circumference in a circle?*” Your Grace will also observe, that all these gentlemen agree in *assuming*—and found their arguments upon the *assumption*—that our Mathematical Tables of sines, co-sines, &c., are strictly correct to 7 places of decimals. But they do more! They *assume* that natural sines as given in Tables are lengths; for if not, how happens it that Mr. Skeat

fancies he proves the perimeter of a polygon of 18 sides to be greater than 3·125, by multiplying by 18 the natural sine of an angle of 10° as given in Tables? Your Grace will not be deceived by this, for you are familiar with the fact known to all *Trigonometers*, that “*the trigonometrical functions of angles are not lengths, but ratios of one length to another.*”

Now, it may be admitted that what are called the natural sines of angles as given in our Mathematical Tables, are calculated from the sine of an angle of 30° to a circle of radius $1 = \frac{1}{2} = \cdot 5$ — and our *Mathematical authorities* will have none other. It may also be admitted, that an angle of 30° is the acute angle of a right-angled triangle, of which the hypotenuse and shortest side are in the ratio or proportion of 2 to 1; and consequently, the hypotenuse being given, the values of all the sides are ascertainable by common Arithmetic.

Now, in decided opposition to all our great *Mathematical authorities* I maintain two things. First:—That the 47th proposition of Euclid’s first book is inapplicable (directly) to the measurement of a curvilinear figure; but I do not dispute that this proposition treats of, and establishes beyond the possibility of dispute or cavil, the properties of a certain rectilinear figure; in other words, I admit that by this proposition, it is demonstrated that in every right angled triangle, the square on the side which subtends the right angle, is equal to the sum of the squares on the other two sides. Second:—That there is a fallacy in the general rule adopted by *Mathematicians* for finding the arithmetical values of the sines, co-sines, &c., of angles: and that, consequently, these values as given in our Mathematical Tables, are for the most part erroneous. I am sure I shall have no difficulty in convincing your Grace on these two points.

Well, then, for this purpose, permit me to direct the attention of your Grace to the enclosed diagram, of which the following may be taken as the construction.

Let AB be a straight line bisected at C , and with C as centre, and CA or CB as interval, describe the semi-circle ADB , and draw the radius CD at right angles to AB , producing the quadrants CDA and CDB . Bisect CD at O , and with O as centre, and OC or OD as interval, describe the circle X , and draw the diameter EF at right angles to CD , dividing that circle into four quadrants. With E as centre, and EO as interval, describe the circle Y . Divide CD into two parts, GC and GD , making GC and GD in the ratio or proportion of 7 to 1, and join EG , producing the right-angled triangle EOG , of which the sides EO and OG must of necessity be in the ratio or proportion of 4 to 3. From the angle G raise the perpendicular GK , making $GK = GD$, and join KC , producing the right-angled triangle KGC . From the point D draw a tangent to the circle X to meet CK produced at the point L , describing the right angled triangle LDC , which is obviously a similar triangle to the triangle KGC . Produce DL to a point M making $DM = 7(DG) = GC$, and join MG . On MG describe the square $MGNP$, and on KC describe the square $KCTR$. From the point A draw a straight line to touch the angles P and R and meet DM produced at V , describing the square $ACDV$, which is a circumscribing square to the circle Y . The sides of the square $MGNP$ cut the circumference of the circle Y at the points m, n, o , and p . Join mn, no, op , and pm , producing the square $mno p$, which is an inscribed square to the circle Y . The circumference of the circle Y is cut by the sides of the square $MGNP$ at four other points, which may be joined,

and produce a second inscribed square to the circle Y . This square I have indicated by dotted lines. On CD describe the square $CDWB$ producing the parallelogram $AVWB$, which is a circumscribing rectangle to the semi-circle ADB . It is obvious that by joining EC , CF , FD , and DE we should obtain an inscribed square to the circle X ; and it is equally obvious that if OE be produced to meet the circumference of the circle Y , the produced line will be a diameter of the circle, and by drawing another diameter at right angles and joining the extremities of the two diameters, we should obtain a third inscribed square to the circle Y . I have not drawn these squares, to avoid confusion in the diagram.

It will be obvious to your Grace, that with F as centre and FO as interval, we might inscribe a circle within the square $CBWD$ exactly equal to the circles X and Y , and then duplicate every geometrical figure within and without the square $ACDV$. It will be equally obvious to your Grace that on the opposite side of the line AB we might construct a similar geometrical figure to that so produced, and thus duplicate every line, circle, triangle, and rectangle, of which the derived geometrical figure would be composed— AB the generating line of the diagram excepted—and we should thus obtain a square on VW a side of the parallelogram $AVWB$, which would be a circumscribing square to a circle—which we may call Z —of which AB the generating line of the diagram would obviously be the diameter, and the arc ADB the semi-circumference: and every circle, triangle, and square, of which the diagram would then be composed, would be within the square on VW the circumscribing square to the circle Z —this square of course, excepted. Then: If RK a side of the square $RKUT$ be produced

to H , as indicated by dotted lines, and a square described on this line, the square $R K C T$ would be quadruplicated, and the square on $R H$ would stand on the circle Z , as the square $P N G M$ stands on the circle Y , but at different angles. I may here remark, that although $A B$ is the generating line, the semi-circle $A D B$ is in fact the generating figure of the diagram.

Professional Mathematicians, such as Professor de Morgan, Mr. G. B. Airy (*the Astronomer Royal*), and others of equal reputation, *audaciously assert*, that no definite relations do, or can exist, between a circle and other geometrical figures, and unhesitatingly put down every man a fool who dares to differ from them. Do these gentlemen really believe what they say? I doubt it! Is it not far more likely, that they are a "*secret confraternity of men*," banded together to make a "*mystery*" of their profession, and "*jealously*" guard it? * If such a geometrical figure as I have described can be constructed, their *assertions* are absurd! If it cannot be constructed, let them prove it!

The impossibility of directing attention to all the properties of this remarkable geometrical figure within the limits of a letter, will be apparent to your Grace; nor is it necessary for my present purpose. I shall confine myself to proving *two* things by means of it. First: That circles and squares of equal superficies may and do exist, and can be isolated and exhibited. Second: That our Mathematical Tables, which Mathematicians *assert* are "*as fixed and certain as the best Interest Tables*," are fallacious, and require rectification.

Now, the square of any binomial = the sum of the

* See the late Prince Consort's Presidential Address at the Aberdeen Meeting of "The British Association."

squares of its two terms, together with twice their product.

Referring your Grace to the diagram, you will observe that CG and GD are the two terms of a binomial; therefore, $CG^2 + 2(CG \cdot GD) + GD^2 = CD^2 =$ area of the square $ACDV$ circumscribed about the circle Y . But, $GK = GD$, by construction, therefore, $GC^2 + GD^2 = GC^2 + GK^2$, and these equations = area of the circle Y ; and, $GC^2 - GD^2 = GC^2 - GK^2 = 6(OE \times 2DG)$ and all these equations = area of a regular inscribed dodecagon or 12 sided polygon, to the circle Y . Now, the 5 coloured right angled triangles KGC , GCN , NAP , PVM , and MDG , within the square $ACDV$, are similar and equal right angled triangles, by construction. Take one of them, say the triangle MDG . Then: MD and DG are equal to the two terms of the binomial $(CG + GD)$, and CG and GD are in the ratio or proportion of 7 to 1, by construction; and $DO = EO =$ radius of the circle Y , by construction. Hence: If $GD = 1$, $MD = 7$, $EO = 4$, and $OG = 3$; therefore, EOG is a right angled triangle, of which the sides EO and OG which contain the right angle, are in the ratio or proportion of 4 to 3. Now, $MD^2 + DG^2 = KG^2 + GC^2 = EO^2 + OG^2 + EG^2 = 3 \cdot 125 (EO^2) = 12 \cdot 5 \{2(DG^2)\} = 50 (DG^2)$, and all these equations = area of the squares $PNGM$ and $RKCT$. But, $25(CD) = 200(GD) = 3 \cdot 125(CD^2)$, and makes 8 circumferences of the circle X or $Y = 25$ diameters, and the circles and squares exactly equal in superficial area. This fixes $\frac{25}{8} = 3 \cdot 125$ as the true arithmetical value of π ; and makes $\frac{1}{3 \cdot 125}$ the true expression of the ratio of diameter to circumference, in every circle.

First proof: let the area of the equal squares $RKCT$

and $PNGM$ be represented by any arithmetical quantity, say 60, and be given to find the values of the sides of the right angled triangle EOG , and the circumference of the circle Y .

Then: $\sqrt{\left(\frac{60}{3 \cdot 125}\right)} = \sqrt{19 \cdot 2} = EO$; $\frac{3}{4}(\sqrt{19 \cdot 2}) = \sqrt{10 \cdot 8} = OG$; and $\frac{1}{2}(\sqrt{19 \cdot 2}) = \sqrt{30} = EG$; therefore, $(EO^2 + OG^2 + EG^2) = (19 \cdot 2 + 10 \cdot 8 + 30) = 60$, is the sum of the areas of squares about the right-angled triangle EOG , and exactly equal to the given area of the squares $RKCT$ and $PNGM$. But, EO is the radius of the circle Y , and $2\pi(r) =$ circumference in every circle, and the product of circumference and semi-radius = area in every circle. Now, $2\pi(EO) = 6 \cdot 25(\sqrt{19 \cdot 2}) = \sqrt{750} =$ circumference of the circle Y ; and (circumference \times semi-radius) = $\sqrt{750} \times \frac{1}{2}(EO) = \sqrt{750} \times \frac{1}{2}(\sqrt{19 \cdot 2}) = \sqrt{(750 \times 4 \cdot 8)} = \sqrt{3600} = 60 =$ area of the circle Y ; and is exactly equal to the area of the squares $RKCT$ and $PNGM$. This makes $12 \cdot 5 \left\{ \frac{1}{2}(EO^2) \right\} = 50(GD^2) =$ area of the circle Y , and proves that $12\frac{1}{2}$ times the area of a square on the semi-radius = area in every circle. Will Professor de Morgan dare to tell me, that the expression $12 \cdot 5 \left(\frac{\pi}{50} \right)$ does not represent the area of a circle of diameter unity, whatever be the value of π ? It is not as a joke I put this question! That gentleman once took in earnest what I *meant* as a joke, and then worked up his imagination into the belief of the *fancy*, that—with reference to the controversy between us—"the best of the argument was in his jokes, and the best of the joke in my arguments." Let Professor de Morgan try to "*retrovert a quip*" and

perpetrate another joke, by attempting to produce the foregoing results with the "mysterious" $\pi = 3.14159265\dots$ and where will he be? The learned Professor has answered this question in the *Athenæum* of July 29, 1865. (*Article: Budget of Paradoxes.*)

Second Proof: Let any arithmetical expression represent the value of GD , say $\sqrt{60}$, and be given to find the area of the circle Y , and the areas of the squares $RKCT$ and $PNGM$.

Then: $25 (GD) = 25 (\sqrt{60}) = \sqrt{37500} =$ circumference of the circle Y ; and $2 (GD) = 2 (\sqrt{60}) = \sqrt{240} = \frac{1}{2} (EO) =$ semi-radius of the circle; and, (circumference \times semi-radius) = area in every circle; therefore, $\sqrt{37500} \times \sqrt{240} = \sqrt{(37500 \times 240)} = 12.5 \{2 (GD^2)\} = 3000 =$ area of the circle Y . Again: $\sqrt{\left(\frac{\text{area}}{\pi}\right)} = 4 (GD)$, that is, $\sqrt{\left(\frac{3000}{3.125}\right)} = 4 (\sqrt{60}) = \sqrt{960} = EO$ the perpendicular of the right angled triangle EOG , and radius of the circle Y . $\frac{3}{4} (\sqrt{960}) = \sqrt{540} = OG$ the base of the triangle EOG , and $\frac{5}{4} (\sqrt{960}) = \sqrt{(\sqrt{960}^2 + \sqrt{540}^2)} = \sqrt{(960 + 540)} = \sqrt{1500} = EG$ the hypotenuse; therefore, $(EO^2 + OG^2 + EG^2) = (960 + 540 + 1500) = 3000 =$ area of the circle Y . But, $KG = GD$, and $GC = 7 (GD)$, by construction; therefore, $KG = \sqrt{60}$, and $GC = 7 (\sqrt{60})$, $= \sqrt{2940}$, and KGC is a right angle; therefore, $KG^2 + GC^2 = (60 + 2940) = 3000 = KC^2$. But, KC is a side of the square $RKCT$, therefore, the area of the square $= 3000$, and the squares $RKCT$ and $PNGM$ are equal; therefore, the area of the circle Y and the area of the squares are exactly equal. *Q. E. D.*

Hence: The area of every circle is equal to the area

of a square on the hypotenuse of a right angled triangle, of which the sides containing the right angle are in the ratio or proportion of 7 to 1, and the sum of these two sides the diameter of the circle.

Corollary: The area of every circle is equal to the sum of the areas of squares about a right angled triangle, of which the sides that contain the right angle are in the ratio or proportion of 4 to 3, and the longer of these sides the radius of the circle.

Professor de Morgan has been pleased to say of me, in the article previously referred to, in a foot note: "Mr. Smith is not mad. Madmen reason rightly upon wrong premises: Mr. Smith reasons wrongly upon no premises at all." And again: "Mr. J. Smith's book shews how a practised arithmetician, venturing into the field of mathematical demonstration, may shew himself utterly destitute of all that distinguishes the reasoning geometrical investigator from the calculator." I may, perhaps, be disposed to admit the assertions of the learned Professor to be true, if he will only be pleased to tell us what is the radius and circumference of the circle Y , when area = 60; if not $\sqrt{19\cdot2}$ and $\sqrt{750}$. Surely a *Mathematician* of his *world-wide* reputation is competent to this!!

It would carry me beyond the limits of a pamphlet suitable for distribution at the forthcoming meeting of the *British Association*, or I might give your Grace proof after proof, by very simple additions to the diagram. I shall content myself by throwing out hints as to some of these methods of proof, which any reader moderately versed in Geometry may readily work out for himself.

Let $(CP + PD)$ denote a binomial of which its two terms CP and PD are in the ratio or proportion of 3 to 1, and the sum of the two terms equal to the line CD in

the diagram. It is obvious that CP will be equal to $CO + \frac{1}{2}(OD)$. With V as centre, and VA or VD as interval, describe a second quadrant within the square $ACDV$, and draw AD the diagonal of the square, dividing the figure formed by the arcs of the quadrants into two equal parts. These additions to the diagram bring into play the algebraical formula: "*The product of the sum and difference of any two quantities = the difference of their squares.*" Now, let a denote the superficial area contained by the arcs of the two quadrants. Let b denote the area of the inscribed square, and c the area of the circumscribing square to the circle Y . Let d denote the difference between the area of circle Y and the area of its inscribed square, and e the difference between the area of the circle Y and the area of its circumscribing square. Let f denote the difference between d and e . Then: $CP^2 = a$. The sum and difference of CP and PD , or the difference of their squares $= b \cdot CP^2 + 2(CP \cdot PD) + PD^2 = e$. $PD^2 = f$; and, $\frac{1}{2}a = d =$ the sum of the areas of the 4 right angled triangles about the square $mno p$. Hence: $b + \frac{1}{2}a =$ area of the circle Y .

Again: we may inscribe 4 isosceles triangles within the square $ACDV$, exactly equal in superficial area to the 4 right angled triangles about the square $PNGM$, each to each. It is self evident that we may produce EO the radius of the circle Y to meet the circumference, and draw another diameter of the circle at right angles to EO ; and it will be obvious to any Geometer, that we may then construct and exhibit 7 more right angled triangles similar and equal to the triangle EOG , and by joining the angles of the 8 triangles, produce 4 isosceles triangles and 4 half squares at the corners of the square $ACDV$. The 4 isosceles triangles will be exactly equal in superficial

area to the 4 right angled triangles about the square $PNGM$. Now, let a denote the sum of the areas of the 8 similar and equal right angled triangles. Let b denote the sum of the areas of the 4 half squares at the corners of the square $ACDV$. Let c denote the sum of the areas of the 4 isosceles triangles. Then : a = area of an inscribed regular dodecagon to the circle Y : $a + b$ = area of the circle Y : and area of the circle $Y + c$ = area of the square $ACDV$ circumscribed about the circle Y . When my friend Mr. Gibbons reads this, he will think of the little corner half squares in the three coloured diagrams, which he found so troublesome to deal with in the merry month of May, 1866 ; but he had a ready way of getting out of a difficulty. It was always either "*I don't see*" or "*I can't see*," and what could I do? He knew I could not prove a negative, and will remember that this led to—my oft repeated expression:—" *If you can't see, I can't help it, but the fact remains notwithstanding.*"

Your Grace must not imagine on this account that Mr. Gibbons is not a *Mathematician*, for I can assure your Grace that I dare back him against Prof. de Morgan any day: but unfortunately, he is like the rest of the confraternity, his head is "*so full of prejudice engendered by crammed erudition, that there is not left a cranny hole for reasoning to get in at.*"

It is not necessary to say more on the first head, and I shall now proceed to prove the second: viz. That there is a fallacy in the method adopted by Mathematicians for finding the arithmetical values of the sines, co-sines &c., of angles; and consequently, that these values as given in tables are for the most part erroneous.

I will at once state the error into which *Mathematicians* have fallen with reference to tables. The natural sine of an angle of 30° to a circle of radius 1 is the same as the

trigonometrical sine of that angle. *Mathematicians* make an angle of 30° to a circle of radius 1 their base, or starting point, in the calculations for tables; and commit the mistake of *assuming*, that because the natural and trigonometrical sine is the same in this angle, the natural and trigonometrical sines are the same in all angles, and, consequently, draw no distinction between them. Of this fact I beg to furnish your Grace with a very distinct proof. Mr. Alex. Edw. Miller of Lincoln's-inn—a *very high class Mathematician*—was one of my public opponents in the *Correspondent*, and the following is an extract from one of his Letters which appeared in that Journal of Jan. 6, 1866. “Mr. Smith's remarks scarcely deserve a reply, and I only offer one lest some non-Mathematical reader should think they do not admit of one. First: ‘*There is no distinction between the trigonometrical and natural sines of angles.*’ Trigonometry may be defined as ‘*the art of using the geometrical ratios,*’ of which I need not say the sine is the principle. Mr. Smith is, I suppose, referring to the distinction between the natural and *logarithmic* sines, &c., which I presume he has seen in the tables without quite comprehending; the one being merely the arithmetical values of the different sines, the other the logarithms of those values.” Even without this piece of evidence from a gentleman of Mr. Miller's known *mathematical* reputation, I am sure it will be within your Grace's own knowledge, that I am right in saying: *Mathematicians make no distinction between the natural and trigonometrical sines of angles.*

Now, the geometrical sine of an angle is half the chord of twice the angle; or, in other words—as my friend Mr. Gibbons puts it—the sine of an arc is half the chord of twice the arc. Hence, in the geometrical operation of

doubling the sides of a polygon within a circle, the ratio of sine to arc must be a *varying* ratio at every step. I shall prove that from either the geometrical, natural, or trigonometrical sine of an angle, we may obtain logarithms by which we can demonstrate the ratio between the sides that contain the right angle, in a right angled triangle; but, I shall also prove that it is only by logarithms obtained from the *trigonometrical* sine, that we can, from the angles and a given side, find the true length of the other two sides.

For this purpose, I must again refer your Grace to the diagram, and direct your attention to the right angled triangle KGC , which I have proved to be similar and equal to the 4 right angled triangles about the square $PNGM$. Now, if CD , the diameter of the circle $X = 1$, $GD = \frac{1}{8} (1) = \cdot 125$, and $KG = GD$, by construction; therefore, $\frac{7}{8} (1) = \cdot 875 = GC$, and GC is common to the binomial $(CG + GD)$, and the right angled triangle KGC . Hence: $(KG^2 + GC^2) = (\cdot 875^2 + \cdot 125^2) = \cdot 78125 = KC^2$, and is equal to the area of the circle X , that is, equal to the area of a circle of diameter unity: and, $4(KC^2) = 3\cdot 125 =$ area of a circle of radius 1; that is, = area of a circle of which AB —the generating line of the diagram—is the diameter.

Now, let the binomial $(CG + GD)$, which is the radius of the semi-circle $ADB = 1$. Then: $\frac{1}{8} (1) = \cdot 125$, = KG , and KG is the *geometrical* sine of the angle C . But, $GC = 7(KG)$, therefore, $7(\cdot 125) = \cdot 875$, = GC , and GC is the *geometrical* co-sine of the angle C . The Logarithm corresponding to the *natural* number $\cdot 125$, is $9\cdot 0969100$; and, the Logarithm corresponding to the *natural* number $\cdot 875$, is $9\cdot 9420081$. But, these Logarithms are neither the *natural* nor *trigonometrical* Log.-sin. and Log.-cos. of the angle C ; and yet, by means of them we

can find the ratio between the sides containing the right angle in the triangle KGC , from a given value of the side KC . For example: Let KC the side subtending the right angle in the triangle $KGC = 8000$, and be given to find the ratio between the other two sides, which contain the right angle.

Then:

As Sin. of angle $G = \text{Sin. } 90^\circ$...	Log.	10'0000000
: the given side $KC = 8000$...	Log.	3'9030900
:: Sin. of angle C	...	Log.	9'0969100
			<hr/>
			13'0000000
			<hr/>
			10'0000000
			<hr/>
: the required side $KG = 8000 (\cdot 125) = 1000$.		Log.	3'0000000
			<hr/>

Again:

As Sin. of angle $G = \text{Sin. } 90^\circ$...	Log.	10'0000000
: the given side $KC = 8000$...	Log.	3'9030900
:: Sin. of angle K	...	Log.	9'9420081
			<hr/>
			13'8450981
			<hr/>
			10'0000000
			<hr/>
: the required side GC			
$= 8000 (\cdot 875) = 7000$...	Log.	3'8450981
			<hr/>

We thus obtain the true ratio between KG and GC , the sides that contain the right angle; that is to say, $1000 : 7000 :: \cdot 125 : \cdot 875$. But, $(1000^2 + 7000^2)$ is not equal to a square on KC , but equal to $3 \cdot 125 \left(\frac{8000}{2}\right)^2$, that is, equal to the area of a circle of which the diameter is 8000.

Now, I have proved, that when the binomial $(CG + GD) = 1, KC^2 = \cdot 78125$; therefore, $\sqrt{\cdot 78125} = \cdot 8838834...$

= $K C$. Then: $\frac{KG}{KC} = \frac{.125}{.8838834} = .14142136 = \sqrt{.02} =$ the *trigonometrical* sine of the angle C ; and $\frac{GC}{KC} = \frac{.875}{.8838834} = 7(\sqrt{.02}) = .98994952 =$ the *trigonometrical* co-sine of the angle C . The Logarithm corresponding to the natural number $.14142136$, is 9.1505150 , and this is the *trigonometrical* Log.-sin. of the angle C . The Logarithm corresponding to the natural number $.98994952$, is 9.9956130 , and this is the *trigonometrical* Log.-cos. of the angle C , and Log.-sin. of the angle K .

Well, then, let KC the side subtending the right angle in the triangle $KGC = 8000$, and be the given side to find the ratio between the other two sides.

Then :

As Sin. of angle $G = \text{Sin. } 90^\circ$...	Log.	10.0000000
: the given side $KC = 8000$...	Log.	3.9030900
:: Sin. of angle C	...	Log.	9.1505150
			<hr/> 3.0536050
			<hr/> 10.0000000

: the required side $KG =$

$$8000 (.14142136) = 1131.37088 \quad \text{Log. } \underline{\underline{3.0536050}}$$

Again :

As Sin. Angle $G = \text{Sin. } 90^\circ$...	Log.	10.0000000
: the given side $KC = 8000$...	Log.	3.9030900
:: Sin. of angle K	...	Log.	9.9956130
			<hr/> 13.8987030
			<hr/> 10.0000000

: the required side GC

$$= 8000 (.98994952) = 7919.59616 \quad \text{Log. } \underline{\underline{3.8987030}}$$

Thus, by different Logarithms we obtain the same result as regards the ratio between the sides KG and GC , in the right angled triangle KGC ; but, the sum of their values as obtained in the latter example, is no longer equal to the diameter of a circle of which $3\cdot125 \left(\frac{8000}{2}\right)^2$ is the area !

This affords no proof whatever of the value of the angle C in the triangle KGC , and I may assume your Grace to put the question :—What, then, is the value of the angle C ? My answer is :—The acute angle of a right angled triangle, of which the sides that contain the right angle are in the ratio or proportion of 7 to 1, is an angle of $80^\circ 8'$. The *geometrical* sine of this angle is $\cdot125$ to a circle of radius 1, and the *trigonometrical* sine of the angle is $\sqrt{\cdot02} = \cdot14142136\dots$ Your Grace might then ask me for my proof. As an honest geometrical controversialist, I am bound to give it !

Well, then, if your Grace have done me the honour to read the Pamphlet I distributed at the Oxford meeting of the *British Association*, in 1860, you will have noticed that in that Pamphlet, I have propounded a theory for finding commensurable right angled triangles; and have proved that from any two given numbers we may find such a triangle ! But, I have also proved that if the given numbers to find a commensurable right angled triangle be consecutive numbers, the difference between the hypotenuse and the longest of the two sides containing the right angle, is a constant quantity = 1. Now, let 1 and 2 be given numbers to find a commensurable right angled triangle. Then : The sum of 1 and 2, or, the difference of their squares = 3 ; twice the product of 1 and 2, = 4 ; and $4 + 1 = 5$: and a triangle of which the

sides are 3, 4, and 5 exactly, is a commensurable right angled triangle. This triangle is the smallest triangle of which the arithmetical values of the sides can be expressed in whole numbers, and I designate it the *primary* commensurable right angled triangle.

Now, let the triangle EOG in the diagram represent the *primary* commensurable right angled triangle. Then: $\frac{OG}{EG} = \frac{3}{5} = \cdot 6$, = sine of angle E ; and $\frac{EO}{EG} = \frac{4}{5} = \cdot 8$, = co-sine of angle E ; and $\cdot 6^2 \times \cdot 8^2 = \text{unity}$, and meets the requirement of the trigonometrical axiom, $\text{Sin.}^2 + \text{Cos.}^2 = \text{unity}$ in every right angled triangle. Hence, the *natural* sine and *trigonometrical* sine of the angle E are the same; and so, of every commensurable right angled triangle. The Logarithm corresponding to the natural number $\cdot 6$ is 9.7781513, and this is the Log-sin. of the angle E , and Log-cos. of the angle G . The Logarithm corresponding to the natural number $\cdot 8$ is 9.9030900, and this is the Log-cos. of the angle E , and Log-sin. of the angle G . This makes the angle E an angle of $36^\circ 52'$, and G is the right angle; therefore, $90^\circ - 36^\circ 52' = 53^\circ 8' = \text{the angle } G$. Proof: Let EG the side subtending the right angle in the triangle EOG , be 8000 miles in length, and be the given side to find the length of the other two sides, and the ratio of side to side.

Then:

As Sin. of angle $O = \text{Sin. } 90^\circ$...	Log. 10.0000000
: the given side $EG = 8000$ miles	...	Log. 3.9030900
:: Sin. of angle $E = \text{Sin } 36^\circ 52'$...	Log. 9.7781513
		13.6812413
		10.0000000
: the required side OG		
= 8000 ($\cdot 6$) = 4800 miles		Log. 3.6812413

Again :

As Sin. of angle $O = \text{Sin. } 90^\circ$...	Log.	10'0000000
: the given side $EG = 8000$ miles	...	Log.	3'9030900
: : the Sin. of angle $G = \text{Sin. } 53^\circ 8'$...	Log.	9'9030900
			<hr/>
			13'8061800
			10'0000000
			<hr/>
: the required side EO			
$= 8000 (\cdot 8) = 6400$ miles	...	Log.	3'8061800
			<hr/>

Hence : $\sqrt{(OG^2 + EO^2)} = \sqrt{(4800^2 + 6400^2)} = 8000 =$ the given length of the side EG .

And, $OG : EO :: 3 : 4$; that is, $4800 : 6400 :: 3 : 4$.

$OG : EG :: 3 : 5$; that is, $4800 : 8000 :: 3 : 5$.

$EO : EG :: 4 : 5$; that is, $6400 : 8000 :: 4 : 5$.

Thus, in the right angled triangle EOG , we have obtained the *known* ratios of side to side, and the values of all the sides, with perfect accuracy.

By Hutton's Tables :

As Sin. of angle $O = \text{Sin. } 90^\circ$...	Log.	10'0000000
: the given side $EG = 8000$ miles	...	Log.	3'9030900
: : Sin of angle $E = \text{Sin. } 36^\circ 52'$...	Log.	9'7781186
			<hr/>
			13'6812086
			10'0000000
			<hr/>
: required side OG
		Log.	3'6812086
			<hr/>

The natural number corresponding to this Logarithm is greater than 4799'6, and less than 4799'7, and makes the side OG less than its *known* and *indisputable* value,

Again :

As Sin. of angle $O = \text{Sin. } 90^\circ$...	Log.	10'0000000
: the given side $EG = 8000$ miles	...	Log.	3'9030900
: : Sin. of angle $G = \text{Sin. } 53^\circ 8'$...	Log.	9'9031084
			<u>13'8061984</u>
			10'0000000
: the required side EO	Log. <u>3'8061984</u>

The natural number corresponding to this Logarithm is greater than 6400'2, and less than 6400'3, and makes the side EO greater than its *known* and *indisputable* value. Thus, by the calculations from Hutton's Tables, the known and indisputable ratios of side to side in the triangle EOG are *destroyed*; and, I am sure cannot fail to convince your Grace, that our Mathematical Tables are fallacious.

This does not prove the angle C in the triangle KGC to be an angle of $8^\circ 8'$, but it is the first step—and a very essential step—towards it; and I shall now proceed to prove, that the angle OEG in the right angled triangle EOG , and the angle KCG in the right angled triangle KGC , are together equal to half a right angle; which makes the angle C in the triangle KGC an angle of $8^\circ 8'$. Well, then, let 3 and 4, that is, the values of the sides containing the right angle in the *primary* commensurable right angled triangle, be given numbers to find another commensurable right angled triangle. Then: The sum of 3 and 4, or, the difference of their squares = 7; twice the product of 3 and 4 = 24; and $24 + 1 = 25$: and 7, 24, and 25, are the values of the sides of a commensurable right angled triangle; and this triangle may be constructed as an addition to the diagram, in the following way. Produce KG to an imaginary point P , (I am obliged to

adopt this imaginary point, as the area of the sheet giving the diagram does not admit of drawing the line,) making GP equal to 24 times KG , or 6 times EO , and join CP . Then CPK will be an isosceles triangle: CG will be a right line drawn from an angle at the base of the isosceles triangle CPK perpendicular to its opposite side: and CGP will be a right angled triangle, of which the sides are 7, 24, and 25, exactly, when $EO = 4$. Hence: The angle KCG at the base, is equal to half the angle P at the vertex, in the isosceles triangle CPK . For, in every isosceles triangle of which the angle at the vertex is not greater than 60° , if straight lines be drawn from the angles at the base, perpendicular to the opposite sides, the two acute angles so obtained, are together equal to the angle at the vertex. As Euclid had to make all his propositions *general*, he does not shew this; but, any Geometer may prove it, by means of the equilateral and equiangular isosceles triangle, of which the sides are equal to the radius of a circle. Hence: If a straight line be drawn from the angle K to a point N , perpendicular to the side CP in the isosceles triangle CPK , the angle CKN will be equal to the angle KCG , and both will be equal to half the angle P , at the vertex of the isosceles triangle CPK .

Now, CGP is a commensurable right angled triangle of which the sides are 7, 24, and 25, exactly. Then: $\frac{CG}{CP} = \frac{7}{25} = \cdot 28$ is the *natural* sine of the angle P ; and $\frac{GP}{CP} = \frac{24}{25} = \cdot 96$ is the co-sine of the angle P ; and $CG^2 + GP^2 = \cdot 28^2 + \cdot 96^2 = \cdot 0784 + \cdot 9216 = \text{unity}$, and meets the requirement of the trigonometrical axiom, $\text{Sin.}^2 + \text{Cos.}^2$

= unity in every right angled triangle. Hence: the *natural* sine and *trigonometrical* sine of the angle P , in the triangle CGP , are the same. The Logarithm corresponding to the natural number $\cdot 28$ is $9\cdot4471580$, and this is the Log.-sin. of the angle P , in the triangle CGP : the Logarithm corresponding to the natural number $\cdot 96$ is $9\cdot9822712$, and this is the Log.-sin. of the angle C , and Log.-cos. of the angle P , in the triangle CGP .

Well, then, let the side CP subtending the right angle G , in the triangle CGP be any length, say 8000 miles; and be the given side to find the length of the other two sides, and the ratios of side to side. Then: The angle P is an angle of $16^\circ 16'$, and G is the right angle; therefore, the angle $C = 90^\circ - 16^\circ 16' = 73^\circ 44'$.

Proof:

As Sin. of angle $G = \text{Sin. } 90^\circ$...	Log.	$10\cdot0000000$
: the given side $CP = 8000$ miles	...	Log.	$3\cdot9030900$
:: Sin. of angle $P = \text{Sin. } 16^\circ 16'$...	Log.	$9\cdot4471580$
			<hr/>
			$13\cdot3502480$
			$10\cdot0000000$
			<hr/>
: the required side CG			
$= 8000 (\cdot 28) = 2240$ miles		Log.	$3\cdot3502480$

Again:

As Sin. of angle $G = \text{Sin. } 90^\circ$...	Log.	$10\cdot0000000$
: the given side $CP = 8000$ miles	...	Log.	$3\cdot9030900$
:: Sin. of angle $C = \text{Sin. } 73^\circ 44'$...	Log.	$9\cdot9822712$
			<hr/>
			$13\cdot8853612$
			$10\cdot0000000$
			<hr/>
: the required side GP			
$= 8000 (\cdot 96) = 7680$ miles		Log.	$3\cdot8853612$

Hence :

$\sqrt{(CG^2 + GP^2)} = \sqrt{(2240^2 + 7680^2)} = 8000 =$ the given length of the side CP .

And, $CG : GP :: 7 : 24$; that is, $2240 : 7680 :: 7 : 24$.

$CG : CP :: 7 : 25$; that is, $2240 : 8000 :: 7 : 25$.

$GP : CP :: 24 : 25$; that is, $7680 : 8000 :: 24 : 25$.

Thus, in the triangle CGP , we obtain the *known* ratios of side to side, and the values of all the sides, with perfect accuracy, and proves that the angle P is an angle of $16^\circ 16'$.

By Hutton's Tables :

As sine of angle $G =$ sine of 90°	...	Log.	10.0000000
: the given side $CP = 8000$ miles	...	Log.	3.9030900
:: sine of angle $P =$ sine $16^\circ 16'$...	Log.	9.4473259
			<hr/> 13.3504159
			<hr/> 10.0000000

: the required side CG

$= 2240.8 \dots$ miles Log. 3.3504159

Again :

As Sin. of angle $G =$ Sin. 90°	...	Log.	10.0000000
: the given side $CP = 8000$ miles	...	Log.	3.9030900
: : Sin. of angle $C =$ Sin. $73^\circ 44'$...	Log.	9.9822569
			<hr/> 13.8853469
			<hr/> 10.0000000

: the required side $GP = 7679.7 \dots$ miles Log. 3.8853469

Thus, by the calculations made from Hutton's Tables, we have destroyed the *known* and *indisputable* ratios of side to side in the triangle CGP ; which again demonstrates beyond the possibility of dispute or cavil, that our Mathematical Tables are fallacious.

Now, the angle C in the triangle $CGK =$ half the

angle P in the triangle $C G P$. I have already shewn that $\sqrt{.02} = .14142136$, is the value of the *trigonometrical* sine of the angle C , in the triangle $C G K$; and I can now give another proof. $C G$ the sine of the angle P , in the triangle $C G P = .28$, and $G K$ the versed sine of the angle $P = K P - G P = 1 - \cos. \text{ of } P = 1 - .96 = .04$; and K is the hypotenuse—or what Mr. Garbett would call the *slant* side—of the right-angled triangle $C G K$; therefore, $(C G^2 + G K^2) = (.28^2 + .04^2) = (.0784 + .0016) = .08 = K C^2$; therefore, $K C = \sqrt{\left(\frac{.28 + .04}{4}\right)} = \sqrt{.08}$. Hence :

The *geometrical* sine and co-sine of the angle C are 1 and 7: the *natural* sine and co-sine are .04 and .28; and $\frac{1}{2}(\sqrt{.08}) = \sqrt{.02} = .14142136$ is the *trigonometrical* sine of the angle C , and equal to the sine of half the angle P ; which makes C an angle of $8^\circ 8'$, and makes the angle E in the triangle $E O G$, and the angle C in the triangle $C G K$, together equal to half a right angle.

Proof: The Logarithm corresponding to the natural number .04 is 8.6020600, and this *we may call* the natural Log.-sin. of the angle C , in the right angled triangle $K G C$: the Logarithm corresponding to the natural number .28 is 9.4471580, and this *we may call* the natural Log.-sin. of the angle K , in the right angled triangle $K G C$: G is the right angle, and *we know* that the sides $G C$ and $G K$ are in the ratio or proportion of 7 to 1, by construction.

Now, let the side $K C$ subtending the right angle, in the triangle $K G C = 8000$, and be the given side to find the ratio between the sides $G C$ and $G K$, which contain the right angle.

Then :

As Sin. of angle $G = \text{Sin. } 90^\circ$...	Log.	10'0000000
: the given side $K C = 8000$...	Log.	3'9030900
:: Sin. of angle C	...	Log.	8'6020600
			<hr/>
			12'5051500
			<hr/>
			10'0000000

: the required side $G K =$

8000 ($\cdot 04$) = 320	Log.	2'5051500
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Again :

As Sin. of angle $G = \text{Sin. } 90^\circ$...	Log.	10'0000000
: the given side $K C = 8000$...	Log.	3'9030900
:: Sin. of angle K	...	Log.	9'4471580
			<hr/>
			13'3502480
			<hr/>
			10'0000000

: the required side $G C =$

8000 ($\cdot 28$) = 2240	Log.	3'3502480
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Thus, $G K : G C :: 1 : 7$; that is, $320 : 2240 :: 1 : 7$; and proves that the sides $G K$ and $G C$ which contain the right angle, in the triangle $K G C$, are in the ratio or proportion of 1 to 7. But, $(G K^2 + G C^2)$ is, *apparently*, not equal to $K C^2$; that is to say, $(320^2 + 2240^2)$ is not equal to $8000^2 = 64000000$. How can this be? Is the 47th proposition of Euclid's first book at fault? Certainly not! It may be asked:—Then how happens it, that $(320^2 + 2240^2) = 51200000$, is not equal to $(G K^2 + G C^2)$? The answer is plain and simple. *The natural and trigonometrical sine of an angle are not the same thing.* There is a distinction between them—notwithstanding the assertion of Mr. Alex. Edw. Miller—which has not hitherto been observed by Mathematicians. Hence: Although

from the Logarithms 8'6020600 and 9'4471580, obtained from the *natural* sine and co-sine of the angle C , we can work out the true ratio between the sides that contain the right angle, in the triangle KGC ; these Logarithms are not the *trigonometrical* Log-sin. and Log-cos. of the angle C . The Log-sin. and Log-cos. of the angle C are 9'1505150 and 9'9956130, as I have already shewn; and I beg to call your Grace's attention to the fact, that the *mantissa* of the Logarithm of the natural number '04 is exactly equal to 4 times the *mantissa* of the trigonometrical Log-sin. of the angle C . *These facts establish the following geometrical and mathematical relations among the various figures in connection with the diagram.* CP the hypotenuse of the triangle CGP = circumference of the circles X and Y . GP the perpendicular of the triangle CGP = the perimeter of a regular inscribed hexagon to the circles X and Y . $2(GP) = GC^2 - GD^2 = GC^2 - KG^2$, and these equations = area of a regular inscribed dodecagon to the circles X and Y . When the triangle EOG represents the *primary* right angled triangle; that is to say, when the sides are 3, 4, and 5, exactly; CG^2 , that is, the square of CG the base of the triangle CGP = the sum of the other two sides; and if we double, or halve, the sides of the triangle CGP , the square of the base increases or diminishes with the sum of the other two sides, in geometrical proportion. The angle OEG in the triangle EOG , plus the angle C in the triangle KGC , are together equal to half a right angle; and twice the angle OEG , plus the angle P in the triangle CGP , are together equal to a right angle. Hence: $\{GC \times \frac{1}{2}(EO)\} = (EO^2 + OG^2 + EG^2) = 3 \cdot 125 (EO^2) = 12 \cdot 5 \left(\frac{EO}{2}\right)^2 = 50 (GD^2) = (CG^2 + GD^2) = (KG^2 + GC^2) = KC^2$; and all these equations = the

area of the circles X or Y , and the area of the squares $R K \hat{C} T$ or $P N G M$. All these facts may be demonstrated in many ways, but to furnish all the proofs would involve other diagrams, and carry me beyond the limits I have prescribed to myself in this letter.

To reasoning and conclusions of the foregoing description, some of my correspondents have answered by saying, "*Stick to Algebra.*" Could anything be more absurd? Can Algebra, *per se*, give us the arithmetical value of anything? The arithmetical value of π is the question in dispute, and it appears to me no greater absurdity is conceivable, than the idea of finding its value by pure Algebra. Such, however, are the extremities to which some of my opponents have been driven; one of them, a teacher of Mathematics in a Collegiate institution. Permit me to put a question to your Grace, which, I am sure you would—on the first blush—answer in the affirmative without the slightest hesitation. Conceive me to give Professor de Morgan the Algebraical formula, $x (C G + G D)^2 = C G + G D$, and tell him that by this formula we can demonstrate the true arithmetical value of π . Would not the learned Professor say I was mad, and would not your Grace agree with him? Now, substitute π for x . Then: It is no doubt true, that as a *general* Algebraical formula, $\pi (C G + G D)^2 = C G + G D$, is absurd; but, as a *particular* Algebraical formula, in connection with the diagram, it is unquestionably true. I have shewn that the *natural* sine and co-sine of the angle C , in the right angled triangle $K G C$, are $\cdot 04$ and $\cdot 28$. Now, if these be the arithmetical values of the two terms of the binomial $(C G + G D)$ in the diagram; then, $3\cdot 125 (C G + G D)^2 = 3\cdot 125 (\cdot 28 + \cdot 04)^2 = 3\cdot 125 (\cdot 32^2) = 3\cdot 125 \times \cdot 1024 = \cdot 32 = C D$, the radius of the semi-circle $A D B$. $2 \pi (C D) = 6\cdot 25 (\cdot 32)$

$= 2 =$ circumference of a circle of which CD is the radius, and AB the diameter; and circumference \times semi-radius $=$ area in every circle; therefore, $2 \times \frac{CD}{2} = 2 \times .16 = \pi (CD^2) = 3.125 (.32^2) = .32 =$ area of the circle of which AB , the generating line of the diagram is the diameter, when CD the radius, $= .32$. Thus, the values of CD and the area of a circle of which CD is the radius, may be represented by the same arithmetical symbols: and proves that $\pi (CG + GD)^2 = (CG + GD)$, when $CD = .32$. Hence, a given quantity may be multiplied by another quantity, without increasing the arithmetical value of the given quantity. I wonder how any of my opponents would prove these facts by Algebra! But further: $4 (CD) = 4 (.32) = 1 \div \frac{\pi}{4} = 1 \div \frac{3.125}{4} = \frac{1}{.78125} = 1.28$; and, 1.28 is the arithmetical value both of the diameter and area of a circle, when the circumference $= 4$. $4 \pi (1.28) = 12.5 \times 1.28 = 16 =$ area of a circumscribing square to a circle, when the diameter $= 4$. But, the area of any circle is found by multiplying the area of its circumscribing square by $\frac{\pi}{4}$; therefore, $16 \left(\frac{\pi}{4} \right) = 16 \left(\frac{3.125}{4} \right) = 16 \times .78125 = 12.5$; and 12.5 is the arithmetical value both of the circumference and area of a circle when the diameter $= 4$. Any Mathematician may readily convince himself, that the values of the circumference and area of a circle are represented by the same arithmetical symbols when the diameter $= 4$, whatever be the value of π ! Hence, 4 is the "*mystic*" number in the Mathematics of Circle-squaring!!

Well, then, the right angled triangle KGC has its sides GK and GC , that is, the sides that contain the right angle, in the ratio or proportion of 1 to 7, by construction. Now, by hypothesis, let C and K be angles of $8^\circ 8'$ and $81^\circ 52'$, together equal to a right angle; and let the side $KC = 8000$, and be the given side to find the ratio between the sides GK and GC by Tables.

Then: by Hutton's Tables:—

As Sin. of angle $G = \text{Sin. } 90^\circ$...	Log.	10.0000000
: the given side $KC = 8000$...	Log.	3.9030900
: : Sin. of angle $C = \text{Sin } 8^\circ 8'$...	Log.	9.1506864
			<hr/> 13.0537764
			10.0000000
: the required side $GK = 1131.8$		Log.	<u><u>3.0537764</u></u>

Again:

As Sin. of angle $G = \text{Sin. } 90^\circ$...	Log.	10.0000000
: the given side $KC = 8000$...	Log.	3.9030900
: : Sin of angle $K = \text{Sin. } 81^\circ 52'$...	Log.	9.9825506
			<hr/> 13.8856406
			10.0000000
: required side $GC = 7684.9$...	Log.	<u><u>3.8856406</u></u>

By these calculations, made from Hutton's Tables, the *known* ratio of GK to GC is *destroyed*. But, my friend Mr. Gibbons would tell me that C is not an angle of $8^\circ 8'$, but an angle that can only be represented by $8^\circ 8' + x$. Let him prove it! I find the *trigonometrical* Log. sines of C and K to be 9.1505150 and 9.9956130, and I have shewn how I get these Logarithms. Will Mr. Gibbons be pleased to point out where I am wrong? Will that gentleman or his friend Professor Adams, who,

I am told, was his college chum, be good enough to tell us, how to prove by Tables, the ratio between the sides that contain the right angle in the triangle KGC , from a given length of KC , the side that subtends the right angle? They may call the angle C either $8^\circ 8' + x$, or $8^\circ 9' - y$, and by the admission of Mr. Gibbons himself, its value is somewhere between $8^\circ 8'$ and $8^\circ 9'$. Your Grace can hardly conceive the shifts to which some of my opponents have been driven, in their attempts to get out of a difficulty. Would your Grace believe the following fact? In one of his letters, Mr. Gibbons boldly asserts, that if all the sides of a right angled triangle can be arithmetically expressed exactly; that is, if the triangle be a commensurable right angled triangle, the angles are inexpressible exactly in degrees and minutes, and *conversely*.

In the next place: The triangles LDC and KGD in the diagram, are similar right angled triangles; and the angle C is common to the two triangles. Euclid proves that $CD : DL :: CG : GK$; and *we know* that CG and GK are in the ratio or proportion of 7 to 1, by construction. Now, by hypothesis, let CD the radius of the semi-circle $ADB = .8$, and represent the semi-radius of a circle. Let Z denote this circle. Then: $12.5 (CD^2) = 12.5 (.8^2) = 12.5 \times .64 = 8 = \text{area of the circle } Z$, and $12.5 (CD) = 12.5 \times .8 = 10 = \text{circumference of the circle } Z$. But, circumference \times semi-radius = area in every circle; therefore, circumference $\times CD = 10 \times .8 = 8 = \text{area of the circle } Z$. But, $\sqrt{\left(\frac{\text{area}}{\pi}\right)} = \text{radius in every circle}$; therefore, $\sqrt{\frac{8}{3.125}} = \sqrt{2.56} = 1.6 = 2 CD = \text{radius of the circle } Z$; and is equal to AB the

diameter of the semi-circle $A D B$, and the generating line of the diagram ; and, $2 \pi (r) =$ circumference in every circle ; therefore, $2 \pi (1.6) = 6.25 \times 1.6 = 10 =$ circumference of the circle Z . Hence : $\pi (1.6^2) = 10 (C D)$ and these equations = area of the circle Z . *These facts establish a distinct relation between the properties of a circle, and our system of Logarithms to the base 10.*

Again : the diagonal of a circumscribing square to a circle of diameter unity $= \sqrt{2}$, and $\sqrt{2}$ appears to have been a stumbling block to Mathematicians in all ages. Now, let squares be circumscribed and inscribed to a circle of radius 1. Then : The area of the circumscribed square $= 4$, and the area of an inscribed square to any circle = half the area of its circumscribing square ; therefore, the area of the inscribed square $= 2$. The side of this square $= \sqrt{2}$, and the perimeter $= 4(\sqrt{2}) = \sqrt{32}$. Now, it is at any rate conceivable, that a circle may exist of which the circumference is equal to the perimeter of an inscribed square to a circle of radius 1 $= \sqrt{32}$; and that such a circle does exist is *Mathematically* demonstrable. For example : Circumference divided by 4 times $\pi =$ semi-radius in every circle ; therefore, $\frac{\sqrt{32}}{4(3.125)} = \frac{\sqrt{32}}{12.5} = \sqrt{\left(\frac{32}{12.5^2}\right)} = \sqrt{\left(\frac{32}{156.25}\right)} = \sqrt{.2048} =$ semi-radius of the circle ; and circumference \times semi-radius = area in every circle ; therefore, $\sqrt{32} \times \sqrt{.2048} = \sqrt{(32 \times .2048)} = \sqrt{6.5536} = 2.56 =$ area of the circle. Now, one-fifth part of the circumference of any circle, is equal to the side of a square containing half the area of the circle.

Proof : $\frac{1}{5} (\sqrt{32}) = \sqrt{\left(\frac{1^2}{5^2} \times 32\right)} = \sqrt{\left(\frac{1}{25} \times 32\right)} =$

$\sqrt{(\cdot 04 \times 32)} = \sqrt{1 \cdot 28}$. Hence : If the side of a square $= \sqrt{1 \cdot 28}$, the area of the square $= 1 \cdot 28$, and is equal to half the area of a circle of which the circumference $= \sqrt{32}$. Again : One-fifth part of the perimeter of any square is equal to the diameter of a circle containing half the area of the square. *Proof*: If the diameter of a circle $= \sqrt{1 \cdot 28}$, then $\frac{1}{2}(\sqrt{1 \cdot 28}) = \sqrt{\left(\frac{1^2}{2} \times 1 \cdot 28\right)} = \sqrt{\left(\frac{1}{4} \times 1 \cdot 28\right)} = \sqrt{(\cdot 25 \times 1 \cdot 28)} = \sqrt{32} =$ radius of the circle ; and $\pi r^2 =$ area in every circle ; therefore, $\pi (\sqrt{32}^2) = 3 \cdot 125 \times 32 = 1 =$ area of the circle, and is equal to half the area of a square of which the perimeter $= 4(\sqrt{2}) = \sqrt{32}$.

Hence : As 4 : 5, so is the diameter of any circle, to the side of a square containing twice the area of the circle. *Corollary* : As 5 : 8, so is the side of any square to the diameter of a circle containing twice the area of the square. *Corollary* : As 4 : 5 :: 5 : 2π . *Corollary* : As 8 : 5 :: 5 : π .

These facts not only establish a distinct relation between the properties of a circle, and our system of Logarithms to the base 10 ; but demonstrate, beyond the possibility of dispute or cavil, that there is a definite relation between the properties of a circle, and the properties of squares. I am sure your Grace will agree with me in the opinion, that if such gentlemen as my friend Mr. Gibbons, Professor de Morgan, and Mr. Airy, the Astronomer Royal, "can't see" these facts, their heads must be "so full of prejudice engendered by crammed erudition, that there is not left a cranny hole for reasoning to get in at."

Again : The line LD in the diagram is a tangent to the circle X , and the triangles LDC and KGC are similar right angled triangles, and have the angle C common to

both. Now, I have proved, that $\cdot 125$ and $\cdot 875$ are the values of the *geometrical* sine and co-sine of the angle C , to a circle of radius 1. I have also proved, that $\cdot 04$ and $\cdot 28$ are the values of the *natural* sine and co-sine of the angle C . But, I have done more: I have proved that the values of the *trigonometrical* sine and co-sine of the angle C , are $\sqrt{\cdot 02} = \cdot 14142136$, and $7(\sqrt{\cdot 02}) = \cdot 98994952$; and the values of the *trigonometrical* Log-sin. and Log-cos. of the angle C , $9\cdot 1505150$ and $9\cdot 9956130$. Now, $7(\sqrt{\cdot 02}) = \sqrt{(7^2 \times \cdot 02)} = \sqrt{49 \times \cdot 02} = \sqrt{\cdot 98}$. But, $\cdot 98^2 = \cdot 9604$, and is a smaller arithmetical quantity than $\sqrt{\cdot 98}$, which is equal to $\cdot 9899495\dots$ This fact is of the utmost importance in the consideration of the questions at issue, and is altogether lost sight of, by our *mathematical Magnates*.

Now, let CD the radius of the semi-circle ADB the generating figure of the diagram = $\cdot 8$. Then: *We know* that the sides containing the right angle in the triangle LDC , are in the ratio or proportion of 7 to 1, by construction; therefore, $LD = \frac{1}{7}(\cdot 8) = \cdot 1142857$, with the recurring decimal 142857 to infinity; that is, with the decimal expression of the fraction $\frac{1}{7}$ to infinity. Now, $\sqrt{(CD^2 + DL^2)} = \sqrt{(\cdot 8^2 + \cdot 1142857^2)} = \cdot 8081220 = LC$, the hypotenuse of the right angled triangle LDC .

Hence: when the side CD in the triangle $LDC = \cdot 8$, $\frac{LD}{LC} = \frac{KG}{KC}$ in the triangle KGC , when $CD = 1$; that is, $\frac{\cdot 1142857}{\cdot 8081220} = \frac{\cdot 125}{\cdot 8838834}$, and these equations = $\sqrt{\cdot 02} = \frac{\sqrt{2}}{10}$; and similarly, $\frac{DC}{LC}$ when $CD = \cdot 8 = \frac{GC}{KC}$ when $CD = 1$; that is, $\frac{8}{\cdot 8081220} = \frac{\cdot 875}{\cdot 8838834} = 7(\sqrt{\cdot 02})$. Thus:

$(.8 \times .8838834) = (.875 \times .8081220) = \sqrt{\{(\frac{7}{10})^2 + (\frac{1}{10})^2\}}$
 $= \sqrt{.7^2 + .1^2} = \sqrt{.49 + .01} = \sqrt{.5} = .7071067\dots =$
 the *trigonometrical* sine of an angle of 45° . These facts not only shew the distinction between the geometrical and trigonometrical sines of an angle, but establish a definite relation between the properties of circles, and the properties of triangles; and again proves that a distinct relation exists between the properties of a circle, and our system of Logarithms to the base 10. The learned Professor de Morgan imagines, that he can stifle truths like the foregoing, by ribald vulgarity. Might I not fairly say to him, as Festus said to St. Paul?—Augustus: “*Thou art beside thyself; much learning doth make thee mad!*”

Once more, and I have done. Professor de Morgan says in his Budget of Paradoxes No. 33:—“*Mr. Smith has quite left off inventing jokes against me, a branch of business in which he seemed to be getting on pretty well for a beginner, when he published his “Nut to Crack.” He now does nothing but pack my own banter—some of it, not all—in marks of quotation, and throw it back at me tail foremost; for he is not yet able to retrovert a quip as it should be done.*” I shall now “*retrovert a quip,*” and shew that the learned Professor “*has convicted himself of ignorance and folly, with an honesty and candour worthy of a better value of π ,*” than 3.14159265 . . .

Well, then, his Budget of Paradoxes No. 27, commences:—“*On June 27 I received a letter, in the handwriting of Mr. James Smith, signed Nauticus;*” and the learned Professor then says:—“*Nauticus lays down—quite correctly—that the sine of an angle is less than its circular measure.*” (True of the geometrical sine of an angle.) “*He (Mr. Smith) then takes 3.1416 for 180° , and finds that 36' is 0.10472. But this is exactly what he finds for the*

sine of 36' in tables : he concludes that either 3.1416 or the tables must be wrong. He does not know that sines, as well as π , are interminable decimals, of which the tables, to save printing, only take in a finite number." (Is the sine of an angle of 30° to a circle of radius 1, an interminable decimal?) "*He is a six figure man : let us go thrice again to make up nine and we have as follows :*

<i>Circular measure of 36'</i>	010471975 ...
<i>Sine of 36'</i>	010471784 ...
<i>Excess of measure over sine</i>	000000191 ...

Mr. Smith invites me to say which is wrong, the quadrature or the tables: I leave him to guess." The learned Professor not knowing the distinction between the *geometrical* and *trigonometrical* sine of an angle, is ignorant of the *fact*, that in small angles the *trigonometrical* sine is arithmetically greater than the *circular* measure of the angle. This I shall now proceed to demonstrate.

Now, I have proved that from any two consecutive numbers, we may obtain a commensurable right angled triangle ; and have directed your Grace's attention to the properties of one of such triangles, of which the sides are 7, 24, and 25, exactly. Well, then, let 24 and 25 be given numbers to find another commensurable right angled triangle. Then : The sum of 24 and 25, or the difference of their squares = 49 : twice the product of 24 and 25 = 1200 : and, $1200 + 1 = 1201$: and 49, 1200, and 1201, are the values of the sides of a commensurable right angled triangle. Let one of the triangles in the diagram denote this triangle, say the triangle KGC . Then $\frac{KG}{KC} = \frac{49}{1201} = .00407993$ is the *trigonometrical* sine of the angle C .

The Logarithm corresponding to the natural number '00407993 is 7·6106527, and this is the *trigonometrical* Log.-sin. of the angle C . $\frac{G}{K} \frac{C}{C} = \frac{1200}{1201} = \cdot 99916736$, and this is the *trigonometrical* co-sine of the angle C ; and—since the sine and co-sine of any angle are the complements of each other—is also the *trigonometrical* sine of the angle K . The Logarithm corresponding to the natural number '99916736 is 9·9996382, and this is the *trigonometrical* Log.-sin. of the angle K . This makes C an angle of $14'$, and K an angle of $89^\circ 46'$. The circular measure of the angle C is $\frac{14' \times \pi}{180 \times 60} = \frac{14 \times 3\cdot125}{10800} = \cdot 004050925$, with the recurring decimal 0925 to infinity, and is *less* than the *trigonometrical* sine of the angle C . "*Nauticus*" may admit, that at the time of writing the letter to which Professor de Morgan refers, he did not know these facts; but that letter proves, that "*Nauticus*" was a sincere and earnest enquirer after scientific truth, and that he was travelling in the right path for reaching it.

Well, then, let the side KC subtending the right angle in the triangle KGC , be 6000 miles in length; and be the given side to find the length of the other two sides, and the ratios of side to side.

Then:

As Sin. of angle $G = \text{Sin. } 90^\circ$...	Log. 10·0000000
: the given side $KC = 6000$ miles	...	Log. 3·7781513
:: Sin. of angle $K = \text{Sin. } 89^\circ 46'$	Log. 9·9996382
		<hr/>
		13·7777895
		10·0000000
		<hr/>
: the required side GC		
= 6000 ($\cdot 99916736$) = 5995·00416 miles.	Log.	<u><u>3·7777895</u></u>

Again :

As Sin. of angle $G = \text{Sin. } 90^\circ$...	Log.	10'0000000
: the given side $KC = 6000$ miles	...	Log.	3'7781513
: : Sin. of angle $C = \text{Sin. } 14'$	Log.	7'6106527
			<hr/> 11'3888040
			<hr/> 10'0000000

: the required side $KG =$

$$6000 (\cdot 00407993) = 24'47958 \text{ miles. Log. } \underline{\underline{1'3888040}}$$

Hence :

$$KG : GC :: 24'47958 : 5995'00416.$$

$$\text{Or, } 49 : 1200 :: 24'47958 : 5994'999 \dots$$

$$KG : KC :: 24'47958 : 6000.$$

$$\text{Or, } 49 : 1201 :: 24'47958 : 5999'995 \dots$$

$$GC : KC :: 5995'00416 : 6000.$$

$$\text{Or, } 1200 : 1201 :: 5995'00416 : 5999'99997 \dots$$

Thus, having obtained the true *trigonometrical* Log sines of the angles C and K , when the right angled triangle KGC denotes a triangle of which the sides are 49, 1200, and 1201, exactly; we find that the ratios of side to side by Logarithms, harmonize with the *known* arithmetical ratios of side to side, by construction. Notwithstanding that we have to deal with incommensurable quantities, it will be obvious to any *geometrical Mathematician*, that although not impracticable to make the approximations arithmetically closer, by extending the number of decimals; for any practical purpose, closer approximations than those obtained by Logarithms to 7 places of decimals, is quite unnecessary. *My clerical opponent calls this cooking a proof by striking off a few figures that displease.* I had adopted this line of argument, and applied it to a triangle of which the sides are 7, 24, and 25, exactly. To this Mr. Gibbons replied:—"Sine $C = \frac{7}{25}$

$= \frac{28}{100} = .28$, but this does not shew $C = 16^\circ 16'$. You must calculate Sine $16^\circ 16'$ and see if it comes out .28 exactly—not look out for a near value in Hutton and then “cook it” by striking off a few figures that displease you. Hutton evidently guided you in assigning the angle, for you offer no computation of it at all.” This is a fair specimen of my friend’s method of evading argument.

I brought these facts under the notice of Professor de Morgan in the following letter, which I addressed to him on the 6th May last :—

My dear Sir,

My last letter afforded you a splendid opportunity of displaying, through the columns of the Athenæum, your talent as a rod-o-mont-ad-ist, of which you have neglected to avail yourself. In a mathematical enquiry there is no occasion for mor-ti-fi-ca-tion—even for a trip—if truth alone be the object aimed at by the enquirers. In the discussion of the questions at issue between us, it is my desire to keep “*within civ-il-i-za-tion*,” and *without* per-turb-a-ti-on ; and with this assurance, so far as I am concerned, I shall end our pen-ta-syl-lab-ic convention.*

Now, Sir, I am going to take a serious part in the game of Mathematics—a game at which two can play—and give you the opportunity of displaying your *moral courage*, by proving to the world that you are an *honest man*, and not a mere *indomitable*

* The reader will see the significance of this and the concluding paragraphs of the letter, by a reference to Professor de Morgan’s supplemental and final Budget, which commences thus :—“*The three paradoxers last named and myself have a pentasyllabic convention, under which, though we go far beyond civility, we keep within civilization. Though Mr. James Smith pronounced that I must be dishonest if I did not see his argument, which he knew I should not do (to say nothing of recent accusation); though Dr. Thorn declared me a competitor for fire and brimstone—and my wife, too, which doubles the joke; though Mr. Reddie was certain I had garbled him, evidently on purpose to make falsehood appear truth; yet, all three profess respect for me as to everything but the power to see truth, or candour to admit it.*” (See “*Athenæum*,” March 30, 1867.)

advocate, playing the part of a defender of mathematical falsehood, and geometrical absurdity.

You, Sir, like my clerical correspondent, maintain that our mathematical tables of sines, co-sines, logarithmic sines, &c., are "*as fixed and certain as our best interest tables.*" So far from this being true, I am about to shew you that, in small angles, these tables lead to the grossest absurdities, and so, "*upset themselves.*"

It is long, Sir, since I first called your attention to the fact that, in small angles some of the natural sines as given in our mathematical tables, are actually greater than the circular measure of the same angles. The circular measure of an angle of $36'$ is, for a "*nine-figure man*" $\cdot 010471975$ as you put it in your "*Budget of Paradoxes*" No. 27, (an *eleven-figure man* would make it $\cdot 01047197531$ with 1 to infinity,) if calculated from $\pi = 3\cdot 1415926$. (See *Athenæum*, 8th July, 1865.) An *eight-figure man* would make it $\cdot 01047196$ with 6 to infinity, if calculated from $\pi = 3\cdot 14159$. A *six-figure man* would make it $\cdot 010472$ exactly, if calculated from $\pi = 3\cdot 1416$, and $\cdot 010416$ with 6 to infinity, if calculated from $\pi = 3\cdot 125$. Hence: 600 times the circular measure of an angle of $36' = 2\pi$, whatever be the value of π . Have you forgotten that this fact you have yourself admitted? If so, permit me to refer you to the *Athenæum*, 5th August, 1865. *Article: Our Library Table.*

You, Sir, agree with my clerical friend, in making an *isosceles* triangle of which the legs $= 1$, and the angle at the apex $= 30^\circ$, your starting point, to find the natural sines of angles of 15° , $7^\circ 30'$, $3^\circ 45'$ and smaller angles. In a right-angled triangle of which the hypotenuse and perpendicular are in the ratio or proportion of 2 to 1, the acute angle is an angle of 30° , and the natural sine of this angle is $\frac{1}{2} = \cdot 5$. Now, Sir, you also agree with my clerical friend in *assuming* that, by calculations based on the peculiar properties of this *particular triangle* only—and that by none other—can we ascertain the sines, co-sines, and Logarithmic sines of angles less than 30° . The validity of this assumption—and I admit that an assumption may be the basis of a sound argument—I *deny*; and maintain on the contrary, that in any commensurable right angled triangle, we can ascertain the angles, sines, co-sines, and Log-sines, with perfect accuracy. And I have shewn you how we may produce commensurable right angled triangles, *ad infinitum*. (See, my letter to you, 17th November, 1866.)

Now, Sir, let the sides KG and GC in a triangle KGC contain a right angle G , and have their arithmetical values represented by 49 and 1200. Then: $KG^2 + GC^2 = 49^2 + 1200^2 = 2401 + 1440000 = 1442401 = KC^2$; therefore, $\sqrt{1442401} = 1201 = KC$ the side subtending the right angle, and the triangle is a commensurable right-angled triangle.

Then: $\frac{KG}{KC} = \frac{49}{1201} = .00407993$ is the natural sine of the angle KCG . The Logarithm corresponding to the natural number .00407993 is 7.6106527, and this is the Logarithmic sine of the angle KCG . $\frac{GC}{KC} = \frac{1200}{1201} = .99916736$ is the natural co-sine of the angle KCG , and—since the sine and co-sine of any angle are the complements of each other—is also the natural sine of the angle CKG . The Logarithm corresponding to the natural number .99916736 is 9.9996382, and this is the Logarithmic co-sine of the angle KCG , and the Logarithmic sine of the angle CKG . This makes KCG an angle of $14'$, and CKG an angle of $89^\circ 46'$, which I shall now proceed to prove.

Well, then, let the side KC subtending the right angle in the triangle KGC be 6000 miles in length, and be the given side to find the length of the other two sides GC and KG .

Then :

As Sin. of angle $KGC = \text{Sin. } 90^\circ$	Log. 10.0000000
: the given side $KC = 6000$ miles...	Log. 3.7781513
: : Sin. of angle $CKG = \text{Sin. } 89^\circ 46'$	Log. 9.9996382
			<u>13.7777895</u>
: the side $GC = 6000 (.99916736)$			<u>10.0000000</u>
$= 5995.00416$ miles	Log. <u>3.7777895</u>

Again :

As Sin. of angle $KGC = \text{Sin. } 90^\circ$	Log. 10.0000000
: the given side $KC = 6000$ miles	Log. 3.7781513
: : Sin. of angle $KCG = \text{Sin. } 14'$	Log. 7.6106527
			<u>11.3888040</u>
: the side $KG = 6000 (.00407993)$			<u>10.0000000</u>
$= 24.47958$ miles...	Log. <u>1.3888040</u>

Hence :

$$\begin{array}{lcl}
 KG : GC :: 24'47958 : 5995'00416 \\
 \text{Or, } 49 : 1200 :: 24'47958 : 5994'999 \dots \\
 KG : KC :: 24'47958 : 6000 \\
 \text{Or, } 49 : 1201 :: 24'47958 : 5999'995 \dots \\
 GC : K^yC :: 5995'00416 : 6000 \\
 \text{Or, } 1200 : 1201 :: 5995'00416 : 5999'99997 \dots
 \end{array}$$

Thus, the ratios of side to side, by Logarithms, harmonize with the ratios of side to side, by construction ; and it is unnecessary for any practical purpose, to make the approximations arithmetically closer by Logarithms.

In your Budget of Paradoxes, No. 27, (see *Athenæum*, 8th July, 1865,) you say :—"He (Mr. Smith) *does not know that sines as well as π , are interminable decimals, of which the tables, to save printing, only take in a finite number.*" There are two falsehoods in this statement. First : π is not an interminable decimal. Second : The angles of 90° , $73^\circ 44'$, $53^\circ 8'$, $36^\circ 52'$, 30° , and $16^\circ 16'$, are exceptions to the rule, and consequently, vitiate your *dogmatical assertion*, that sines—by which, if you mean anything, you mean *all* sines—are interminable decimals ; and you have incontrovertible evidence of these facts in some of my recent letters.

Now, my good Sir, you tell us you have made yourself "*a public scavenger of science,*" and "*look down upon other scavengers,*" such as "*Montucla, Hutton, &c., as mere historical drudges,*" and "*not fit to compete with you :*" and yet, I hardly think you will venture to tell me that, the "*broom*" of Hutton's manufacture is not Orthodox. Now, Sir, let me suppose you to take in hand to sweep round the foregoing scientific "*post*" with Hutton's "*broom.*"

Then :

As Sin. of angle $KGC = \text{Sin. } 90^\circ$	Log. 10'0000000
: the given side $KC = 6000$ miles	Log. 3'7781513
: : Sin. of angle $CKG = \text{Sin. } 89^\circ 46'$	Log. 9'9999964
			13'7781477
			10'0000000
: the side $GC = 5999'9$ miles	Log. 3'7781477

Again :

As Sin. of angle $KGC = \text{Sin. } 90^\circ$	Log. 10'0000000
: the given side $KC = 6000$ miles...	Log. 3'7781513
: : Sin. of angle $KCG = \text{Sin. } 14'$	Log. 7'6098530
			<hr/> 11'3880043
			* 10'0000000
: the side $KG = 24'434$ miles	Log. 1'3880043

So that by working with Orthodox tools, you would destroy the inequalities between the two sides of your "house" KC and GC , get rid of the nook or corner C , swallow up the *trine*, and yet leave the side of your "house" KG upwards of 24 miles in length. In other words: As a *scavenger* with Hutton's "broom" you would sweep into the land of oblivion the axiom in Trigonometry, that "the functions of angles are not lengths, but ratios of one length to another," by destroying the ratios of side to side in a commensurable right-angled triangle, of which the sides are 49, 1200, and 1201, by construction. Could absurdity go farther? I should like to see you go to work again, and try your hand at "sweeping round a paradoxer" that holds on to a genuine scientific "post." To be serious: Have I not demonstrated, my good Sir, that in a triangle of which the sides are 49, 1200, and 1201, the obtuse angle is an angle of $89^\circ 46'$, and the acute angle an angle of $14'$? I await your answer to this plain and simple question!

Now, Sir, in my letter to you of the 26th March, I have proved the following geometrical and mathematical truths; and my proofs cannot be controverted, either by your ingenuity, or that of any other Mathematician.

First: The circular measure of an angle of $15^\circ = \frac{15^\circ \times \pi}{180}$
 $= \frac{1}{3} \left(\frac{\pi}{4} \right)$, whatever be the value of π .

Second: The circular measure of an angle of any number of degrees, divided by the circular measure of an angle of the same number of minutes, is a constant quantity = the perimeter of a regular inscribed hexagon to a circle of radius 1 = 6, and is demonstrable by means of any hypothetical value of π .

Third: The circular measure of a right angle, divided by the circular measure of an angle of $14^\circ 24' = \frac{1}{25} (360^\circ)$, is a constant

quantity, = 6.25, and is demonstrable by means of any hypothetical value of π .

Fourth: An arc equal to radius, subtending an angle at the centre of a circle = $\frac{4}{25}$ (circumference).

Fifth: The area of a circle of radius 1, and circumference of a circle of diameter unity, are represented by the same arithmetical symbols, whatever be the value of π . And, every other value of π but that which makes 8 circumferences of a circle exactly equal to 25 diameters; and $\frac{6}{6.25}$ the true expression of the ratio between the perimeter of any regular hexagon and the circumference of its circumscribing circle; would make the area of a circle of radius 1, either greater or less than π . Hence: *It is impossible, that the true arithmetical value of π can be anything else but $\frac{6.25}{2} = \frac{25}{8} = 3.125$.*

Do you imagine that you can “*mislead the public, and silence truth*” for any length of time, by *affecting to despise* truths like these? Be assured, my good Sir, that the public will ultimately be our judges, and pronounce a just judgment too, on the respective parts you and I have played in the search after, and defence of, scientific truth.

In conclusion: much now depends on the course you may adopt. If you resolve to play a *discourteous* part, and decline to answer the plain and simple question I have put to you, under the vain impression that you are independent of, and superior to, *public opinion*; you may find me a *thorn* (Dr. Thorn)—and a *thorn* more irritating than a barbed arrow—in your side: and you may also find me *ready* (Mr. Reddie) to force it home too; aye, as *ready* as a *smith*, who not only knows how to forge a nail, but knows how to drive it home, and clinch it. I hope, however, to find, that you choose a wiser and better part; and that it may lead to our shortly meeting and shaking hands, in Mathematical fellowship and Christian charity.*

Believe me, my dear Sir,

Very sincerely yours,

JAMES SMITH.

To PROFESSOR DE MORGAN, F.R.S., &c., *London*.

* Professor de Morgan will find this paragraph slightly altered in phraseology; but, if he think fit, the writer has no objection to offer against his publishing the original!

The Professor has never answered my letter privately, or referred to it publicly; and it would almost appear as if the mathematical *hierarch* of the *Athenæum* had been suddenly transformed into a "*dumb dog that cannot bark*," and become incapable of giving an oration from the "*Athenæum pulpit*." Be this as it may, if Professor de Morgan knows not the difference between the *natural* and *trigonometrical* sine of an angle; and knows not that in small angles the *trigonometrical* sine is greater than the *circular measure* of the angle, does he not stand convicted of ignorance? I put this question in a Mathematical sense, but I have no hesitation in telling your Grace, that though the learned Professor *was* ignorant of the distinction between the *natural* and *trigonometrical* sine of an angle, when he wrote his "*Budget of Paradoxes*," he knows that distinction now; aye, and he knows too, that in small angles the *circular measure* is *less* than the *trigonometrical* sine. Does he imagine that by playing the part of a "*dumb dog*" he can preserve and protect his Mathematical reputation? If so, is he not chargeable with folly? If he think that by silence he can "*make falsehood appear truth*," and get all the world to believe him, is he not chargeable with folly? Can he decline to admit his previous ignorance of the facts I have brought under his notice, "*without offence to his own conscience*;" and in tampering with conscience, is he not chargeable with something worse than folly? If he have "*the power to see truth*" but not the "*candour to admit it*," is he not chargeable with something worse than either ignorance or folly? And I put the following question to your Grace:—Has not Professor de Morgan *convicted himself, both of ignorance and folly*, in affording *the writer* the opportunity of putting such questions as these before the public? I cannot help thinking your

Grace will answer this question in the affirmative, without any hesitation !

On the 7th May, I addressed a letter to my correspondent, Mr. Gibbons, which commenced as follows :—"In your letter of the 18th January last, you tell me in the most distinct terms, that I do not understand the construction of the tables of sines and co-sines ; and this expression of opinion you reiterate in your letter of the 20th February. I now send you copy of a letter I posted yesterday to Professor de Morgan, which, if you '*read, mark, learn, and inwardly digest*,' together with what follows it, I cannot help thinking you will modify your opinion as to my knowledge of the tables of sines and co-sines." Then followed copy of my letter of the previous day to Professor de Morgan. I then gave the method of construction of a very remarkable diagram, shewing the geometrical connection between the right angled triangles, of which the sides are 3, 4, and 5, exactly ; and 7, 24, and 25, exactly ; and was about to prove, that when the sides of a triangle are in the ratio or proportion of 3, 4, and 5, the acute angle is an angle of $36^{\circ} 52'$ —which I had promised to do in a previous letter—when a communication of my correspondent's reached me. My letter then proceeds :—"I had written so far, and was about to deal with the equal angles ACB and BCH , when your favour of the 3rd inst. (*posted yesterday*) came to hand, and I must pause to notice it."

"*The postscript to your letter commences with a question :—'If $BC = \cdot 261$, how can $\frac{\pi}{12}$ be less than $\cdot 261$?'—and ends with an exclamation :—'Surely, it is ridiculous to argue further !'*"

"Now, my dear Sir, it is not quite so ridiculous as—at first sight—may appear to you, to *argue a little further*."

I have told you that your "*first erroneous step*" (*Mr. Gibbons had asked me to point out his first erroneous step*) is one of principle, not of calculation: and I shall see how you deal with the arguments, by which I have proved this fact, in my two last letters; the first of which you had obviously not read through; and the latter was not to hand, when you penned the postscript to your letter. You are also labouring under another fallacious principle—which I have pointed out in the foregoing letter to De Morgan—when you ask the question:—"If $BC = \cdot 261 \dots$ how can $\frac{\pi}{12}$ be less than $\cdot 261 \dots$? You forget that the trigonometrical functions of angles—and sines are functions of angles—are *not lengths but ratios of one length to another*. It is by treating sines as *lengths* that leads you to the *fancy*, that I make the perimeter of a 24 sided regular polygon, greater than the circumference of its circumscribing circle."

My letter concluded as follows:—"Now, my dear Sir, hypothetically assume ECD (*the acute angle in a triangle of which the sides are 7, 24, and 25, exactly*) to be an angle of $16^\circ 15'$ or $16^\circ 17'$ —work out the calculations by Hutton's Tables—and this is simple enough. You will then find that, in either case, you *destroy* the known ratios of side to side in the triangle, but in opposite directions; and this demonstrates—*beyond the possibility of dispute or cavil by any honest Mathematician*—that ECD is an angle of $16^\circ 16'$, and DEC an angle of $73^\circ 44'$. I await your next communication."

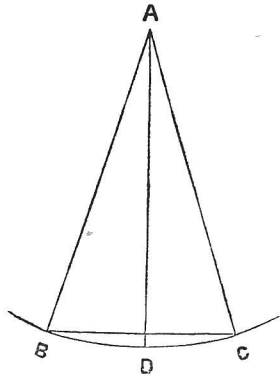
To Mr. Gibbons there was nothing new in my letter to Professor de Morgan; for, in a letter dated 4th May, I had shewn him the effect of Hutton's Tables upon small angles; and every other point referred to in my letter to de Morgan, I had proved in detail in previous

lettersto him. Can Mr. Gibbons point out when, or where, he has ever attempted to grapple with any one of the five facts referred to in my letter to de Morgan? I trow not! As an answer to my letters of the 29th April, and 4th and 7th May, Mr. Gibbons' letter of the 11th May is so *illogical* that I give it in extenso, with a few comments upon it:—

THE REV. GEORGE B. GIBBONS, B.A., to JAMES SMITH, ESQ.
LANEAST, LAUNCESTON,
11th May, 1867.

DEAR SIR,

I have really exhausted all I have to say. $AC = AB = 1$, and $A = 15^\circ$. You may call $\cdot 261$...a ratio or a length as you please, but the meaning is, that if the radius $= 1$, the chord $BC = \cdot 261$...or, $BC : AC :: \cdot 261 : 1$ (a ratio). BC is $\cdot 261$...by the same measurement that gives $\text{Sin. } 30^\circ = \cdot 5$, or, $\text{Sin } 90^\circ = 1$.



This *assertion* appears plausible, but is both fallacious, and indemonstrable. An angle of 90° is contained by two radii of a circle at right angles; and the *geometrical* sines of angles of 60° and 30° (*the one being the complement of the other, the sine of the one is the co-sine of the other*) are perpendicular lines to one of these radii, and parallel lines to the other; and so, of all angles intermediate between 30° and 90° . Hence: To prove his *assertion*, Mr. Gibbons must shew, that the same measurement that gives $\text{Sin. } 30^\circ = \cdot 5$,

and $\text{Sin. } 90^\circ = 1$; makes the sines of 15° and $7^\circ 30'$ perpendicular lines to one radii of the circle, and parallel lines to the other.*

Put generally, arc $BDC = \frac{\pi r}{12}$, and chord $BC = \cdot 261 (r)$. But adopting unity as the radius, $BC = \cdot 261 \dots AC = 1$, whilst $\frac{\pi}{12}$ is the arc to the same measurement. But whatever be the radius, or whether you call BC a length, or a ratio, $\pi = 3\cdot 125$ would make the arc shorter than the chord, and your objection is to me unintelligible. So I beg respectfully to be excused discussing this any further, and confess my utter inability to understand what your objection means.

I had called Mr. Gibbons' special attention to the fact, that the ratio of sine to arc in an angle of 30° , is a varying ratio from sine to arc in an angle of 15° ; or, sine to arc in an angle of $7^\circ 30'$; the *geometrical* sines at every step being divergent lines from the sine of an angle of 30° , and proportionally longer lines. Hence, the ratio, $\frac{\cdot 261 \dots}{2} : 7^\circ 30'$, does not express the true *trigonometrical ratio* of sine to arc in an angle of $7^\circ 30'$.

*A person may be an *expert* in Mathematics, and thoroughly *versed* in all the propositions of Euclid; and yet, not be a practical Geometer. "*I would suggest one little*" proposition, and ask Mr. Gibbons, with the assistance of his friend Professor Adams, the *learned* Professor de Morgan, and Mr. Airy, the Astronomer Royal, to solve it. *From a right angled triangle (as the generating figure) of which the sides are 3, 4, and 5, or in these proportions, exactly; construct a diagram, representing a geometrical figure, which shall contain two dis-similar and unequal right angled triangles; so that $\sqrt{41}$ may represent the arithmetical value of the hypotenuse, in both.* The sides containing the right angle in the one will be represented by $\cdot 4$ and $\cdot 5$, and in the other by $\cdot 38418744$ and $\cdot 51224992$, and in both the hypotenuse $= \sqrt{41}$. As *Mathematicians*, the proof of this will be mere child's play to these gentlemen. *Let them construct the diagram, and prove that they are practical Geometers!!*

"As to Hutton's Tables, as I did not employ them, or any other, in calculating BC ; their errors, if any such exist, would not affect the question. In the triangle of which the sides are 7, 24, and 25, and A the acute angle, $\cos. A = .96$, and $\text{Log. } .96 = 9.982271233032568$, &c.; $\text{Log.-cos. } 73^\circ 44' = 9.9822569 \dots$ so that I don't see how $A = 73^\circ 44'$ exactly, which you assert it to be."

In these remarks Mr. Gibbons *assumes* the infallibility of Mathematical Tables, one of the very questions in dispute.

"Every letter of yours now is so full of assertions that I cannot agree with, and so entirely beyond my comprehension, that you will pardon my not replying. When an opponent writes me:—(*See yours, 29th April*) You are right in saying that $BC = .261 \dots$ when $AB = AC = 1$, and yet asserts that $\frac{\pi}{12} = \frac{3.125}{12} = .2604 \dots$ or the arc less than the chord, I frankly confess he has gone beyond my powers of comprehension. I hope this is the last letter I shall have to write on this tiresome topic, and I think you will allow that I have written enough already."

In my letter of the 29th April, I had called Mr. Gibbons' attention to the five propositions I have referred to in my letter to Professor de Morgan, and then went on to say:—

"These, my dear Sir, are geometrical and mathematical truths, which by no ingenuity of the Mathematician can be controverted; and I cannot but express my surprise that you should think you have got over a difficulty, by a mere reiteration in your letter of the 1st April of the one stale argument, which letter I presume I must accept as your reply to mine of the 25th March.* Well, then, in the face of truths like these, it is worse than absurd to imagine, that you can deduce *by arithmetic process*, from $\text{Sin. } 30^\circ = \frac{1}{2}$, or by any other process, that the perimeter of a regular 24 sided polygon to a circle of radius 1 is *greater* than 6.25."

"You may still ask me to point out '*the first erroneous step in your calculation of $BC = .261 \dots$* ' and *fancy* that

* The letter of March 25th dealt with the five propositions in detail,

until I do so, you are a triumphant opponent 'in a fair *Mathematical fight*.' Well, then, on this supposition I observe:—Your '*first erroneous step*' is one of principle, not of calculation. You make the sine of an angle of $30^\circ = \frac{1}{2} = \cdot 5$, your starting point. Your next step is to find the arithmetical value of the base of an isosceles triangle subtending an angle of 30° . Now, my good Sir, *I must assume* that you proceed to work *geometrically*; and, in the first place, construct the right angled triangle, of which the *natural* sine subtends an angle of 30° ; that is to say, construct a right angled triangle of which the hypotenuse and perpendicular are in the ratio of 2 to 1. This we may call the *generating* triangle. I shall *assume* further—though not absolutely a matter of necessity—that as the readiest way of getting an isosceles triangle of which the base subtends an angle of 30° , you describe a circle, with the acute angle of the *generating* triangle as centre, and hypotenuse as interval; then produce the base of the *generating* triangle to meet the circumference of the circle, and join the extremities of the two radii of the circle thus obtained. The base of this isosceles triangle is the hypotenuse of a right angled triangle, of which the sides containing the right angle, are the sine and versed-sine of the *generating* triangle, and is the chord of an arc of 30° . (*There is no great stretch of imagination, in supposing you to inscribe twelve equal isosceles triangles within the circle, producing a regular inscribed dodecagon; if so, you know as well as I do, that the area of the dodecagon is to the area of the circle, as the perimeter of an inscribed regular hexagon to the circumference of the circle; whatever be the value of π*). Your next step is to find the arithmetical value of the base of an isosceles triangle subtending an angle of 15° . For this purpose you draw a straight line bisecting the angle of 30° at the centre of the circle, and its subtending chord and arc,

and join the extremity of the radius thus obtained, with that extremity of the *natural sine* of the *generating* triangle that touches the circumference of the circle ; producing another isosceles triangle of which the legs are radii of the circle, and one of the legs the hypotenuse of the generating triangle. (*The other leg does not bisect the natural sine of the acute angle of the generating triangle*). The base of this isosceles triangle is the hypotenuse of a right angled triangle, of which the sides containing the right angle, are the sine and versed-sine of an angle of 15° . Your next step would be to find the base of an isosceles triangle subtending an angle of $7^{\circ} 30'$ by the same process, and so on for smaller angles. (*At this point, I must refer you to the diagram on the first page of this letter, and to the paragraph on the third page marked with a bracket, in which I shew another method of bisecting the base of an isosceles triangle, and prove that half the base is a longer line than half the sine of its subtending angle ; which vitiates your assumption, that from the sine of an angle of $7^{\circ} 30'$, you can prove the perimeter of a regular polygon of 24 sides to be greater than the circumference of its circumscribing circle, on the theory that 8 circumferences of a circle are exactly equal to 25 diameters.*) Now, the base of each successive isosceles triangle, is a divergent line from the *natural sine* of the *generating* triangle, and diverges more and more at every step, as we proceed with this geometrical operation ; and while one extremity of the base of the isosceles triangles is fixed, *as it were*, and cannot get away from one extremity of the *natural sine* of the *generating* triangle, the other extremity of the base of the triangles is working round an arc of 30° ; that is to say, is working round an arc subtending the acute angle of the *generating* triangle. But, at the very first step in this operation, the base of the triangle is longer in proportion to its subtending

arc, than sine to arc at your starting point; or in other words, the chord is greater in proportion to its subtending arc in an angle of 30° than in an angle of 60° ; and in an angle of 15° than in an angle of 30° ; and it follows necessarily, that in the calculations by '*arithmetic process*' to a circle of radius 1, we are carried beyond the true circumference of the circle. Hence: the *fallacy* and *absurdity* of your supposing that you can employ the sine of the angle BAD , and prove the perimeter of a regular polygon of 24 sides to be greater than 6.25; that is, greater than the true circumference of the circle when radius = 1. The fact is, as I have said repeatedly, the 47th proposition of the first book of Euclid treats of a *rectilinear* figure, and is inapplicable *directly* as a measure of value of any *curvilinear* figure; but, *indirectly*, it plays a most important part in ascertaining the true ratio of diameter to circumference in a circle."

"I shall receive willingly, and read (till I come to something I cannot understand) anything you may favour me with. If I *do not answer*, it will be because I have already offered all I have to say. Professor de Morgan is far better game for you, and his views and mine are identical. It is time for you to have a new opponent."

Believe me, dear Sir,

Yours truly,

GEO. B. GIBBONS.

So much for the *logic* of the Rev. Geo. B. Gibbons, B.A., of St. John's College, Cambridge. From my experience of Mathematicians—whether professional or non-professional—and I have had no little experience of both, in the past eight or nine years—it would appear as if "*crammed erudition*" in Mathematics, is calculated to produce *obtundity*—if I may be permitted to coin a word—rather than *profundity* of intellect.

Your Grace will not require the assistance of such gentlemen as Mr. Gibbons and Professor de Morgan, to enable you to comprehend what I am about to bring under your notice ; and, having directed your Grace's attention to certain truths with reference to the remarkable and interesting geometrical figure represented by the diagram, I shall have "*offered all I have to say*" with regard to it, for the present.

The square $A C D V$ is a circumscribed square, and the square $m n o p$ is an inscribed square, to the circle Y . The square $P N G M$ is an intermediate square between these circumscribed and inscribed squares to the circle Y , and is inseparably connected with both, by construction. The three squares are produced from, and stand inseparably connected with, the circle Y ; and your Grace cannot fail to perceive, that it is only in consequence of this inseparable connection, and the definite relations existing between the squares and the circle, that the inscribed square $m n o p$ can be geometrically produced and made to occupy the position represented in the diagram. Now, it is obvious that we cannot duplicate a circle within its circumscribed square, or about its inscribed square ; but, we may duplicate the square $P N G M$, and isolate and exhibit the square standing on the circle Y , within the square $A C D V$, in juxtaposition to the square $P N G M$; and we may then describe a circumscribing circle to the two squares. Well, then, let it be required so to duplicate the square $P N G M$. This may be done in several ways, but the following method is the most simple and direct.

Your Grace will observe, that NG the hypothenuse of the right angled triangle NCG , and EO the perpendicular of the right angled triangle EOG , are intersecting lines. Draw a straight line, say AB , parallel to LC the

hypothenuse of the right angled triangle $L D C$, through the point of intersection between NG and EO , to meet VD and CD , sides of the circumscribing square to the circle Y . Then: ADB and NCG will be similar and equal right angled triangles, and a square described on AB will be the required square. The next most direct method of producing the required square is this. From the angles m and p of the inscribed square $mno p$, draw the diagonals of the square, and produce them to meet VD and CD , sides of the circumscribing square to the circle Y , at points A and B . Join AB , and on AB describe a square. We again get the similar and equal right angled triangles ADB and NCG , and the square on AB will be the required square. There are other methods of duplicating the square $PNGM$, equally geometrical, but more complicated, and of course less direct than the foregoing; but it is not necessary to refer to them for my present purpose.

Now, I must assume that your Grace has made the addition of this square to the diagram, and that you have it before you, with a square on AB standing on the circle Y , in juxtaposition to the square $PNGM$. Then: Join the adjacent angles of the two squares, and draw the diagonals, producing four isosceles triangles. If your Grace will be pleased to take the trouble to work out the calculations, you will find that these isosceles triangles are exactly equal in superficial area to the right angled triangles about the squares, each to each. Assuming the diameter of the circle $Y = 8$, the area of each of the isosceles triangles is equal to the area of a rectangle of which the longer side $= \sqrt{24 \cdot 5}$ and the shorter side $= \sqrt{5}$; therefore, $(\sqrt{24 \cdot 5} \times \sqrt{5}) = \sqrt{12 \cdot 25} = 3 \cdot 5 =$ area of one of the isosceles triangles. Hence: The area of the square $ACDV$ circumscribed about the circle Y ,

minus the sum of the areas of the four isosceles triangles
 $= 64 - 14 = (EO^2 + OG^2 + EG^2) = 3.125 (4^2) =$
 $3.125 \times 16 = \pi (EO^2) = 50$, and is therefore equal to the
 area of the circles X and Y , the area of the squares
 $PNGM$, $RKCT$, and the area of the square on AB .

Again: With E as centre, and EG the hypotenuse
 of the right angled triangle $EOG =$ half the diagonal
 of the squares $PNGM$, $RKCT$, or the square on AB ,
 as interval, describe a circle. This circle will be a cir-
 cumscribing circle to the square $PNGM$ and the square
 on AB ; and if the diameter of the circle $Y = 8$, the
 diameter of this circle $= 10$. Hence: The diameter of
 the circles are to each other in the same ratio or propor-
 tion as EO the perpendicular to EG the hypotenuse,
 in the right angled triangle EOG , that is, in the ratio or
 proportion of 4 to 5.

How could facts like these be possible, My Lord, if
 there were no definite relations existing between the
 circles, squares, and triangles, of which the geometrical
 figure represented by the diagram is composed? Well,
 then, I have shewn your Grace that the most perfect
 harmony prevails between Geometry, Trigonometry, and
 Mathematics; and our scientific *Magnates* cannot
 much longer persist in making geometrical truth, involve
 mathematical absurdity.

To you, My Lord, the truths I have brought under
 your notice will be plain and simple enough; and I cannot
 help thinking, that even the marvellous combination of
 scientific intellect represented by *The British Association*
for the Advancement of Science, will not be competent
 to the task of convincing your Grace, that the solution
 of the problem of *The Quadrature of the Circle*, is not
 “*un fait accompli*.”

It is quite possible that a person may be well versed in all the higher branches of Mathematics, such as the differential and integral calculus; and be a thorough master of every proposition in Euclid; and yet, not be a *practical Geometer*. Such are Professor de Morgan and the Rev. Geo. B. Gibbons, B.A. But, it will be as obvious as that 2 and 2 make 4, to any Mathematical reader who is a "*reasoning geometrical investigator*," that the solution of the problem of the Quadrature of the Circle has at length been discovered, and with it, the true solution of a Right Angled Triangle. From these discoveries we find, and in the foregoing pages I have demonstrated, that existing Mathematical Tables are fallacious, and require rectification; and I might go on to prove that the Moon's Horizontal Parallax, as given in the Nautical Almanac, is also erroneous. Hence, the Navigator is unable to fix the true position of his ship at sea, from a lunar or astral observation; and no wonder, when he has to work out his calculations from false meridians of longitude, and fallacious tables. *Who* can tell the loss of life and property that may have been the result? Well, then, I would ask your Grace, does not the consideration of such subjects come legitimately within the professed objects of the British Association? How happens it then, that the "*guiding stars*" of that body refuse to permit the discussion of such subjects in their proceedings? It may be said: We have Astronomers Royal of England, Scotland, and Ireland, paid public servants; and it may be asked:—Is it not more properly a part of *their* duty to enquire into such matters? Probably it may be, but who is to compel them to do it? Your Grace will know better than I do, but it strikes me that this is a matter that properly comes within the province

of the Board of Trade department of the Government. Be this, however, as it may, I am decidedly of opinion, that the subjects I have brought under your Grace's notice, are legitimately within the professed objects of *The British Association for the Advancement of Science*; and that it is with that Association an enquiry into such subjects should originate.

Your Grace will find in the *Athenæum* of Sept. 9, 1865, or, in the Transactions of the Association for that year, Professor Phillip's Presidential Address, which concludes as follows :—

"Here, indeed, is the stronghold of the British Association. Wherever and by whatever means sound learning and useful knowledge are advanced, these to us are friends. Whoever is privileged to step beyond his fellows, on the road of scientific discovery, will receive our applause, and, if need be, our help. Welcoming and joining in the labour of all, we shall keep our place among those who clear the roads and remove the obstacles from the paths of science; and whatever be our own success in the rich fields that lie before us, however little we may now know, we shall prove that in this our day, we knew at least the value of knowledge, and joined hands and hearts in the endeavour to promote it."

This is, indeed, a splendid peroration. Will it bear the light of truth? If so, it would be worthy of Professor Phillips and the *British Association*. I speak from experience when I say :—I have *grave doubts* on this point, and will afford your Grace the opportunity of testing the *honesty* and *sincerity* of Professor Phillips and his compeers. This can be done by your Grace—as President of the Association—affording me the opportunity of reading a paper in the Physical Section, at the forthcoming meeting. Subject: *The true Solution of a*

right angled Triangle, and the Fallacious character of existing Mathematical Tables. I shall be prepared to read such a paper at an hour's notice.

I am an old Life member of the Association, and in times gone by have taken much—and still take some—interest, in its *progress, prosperity, and permanency*. Its very *existence* is now in danger, and for this reason I have felt impelled to address your Grace, as the President elect, on the important questions referred to ; which are legitimately within the province of the *British Association*, and in which not only the interests of science, but the interests of humanity are involved. In so addressing your Grace I have done my duty ; it remains for Professor Phillips and the other “*guiding stars*” of that body to do theirs !

In conclusion : It is refreshing to me to think—and will not be uninteresting to your Grace to hear—of one pleasing incident in the history of my connection with the *British Association*. My Oxford pamphlet led to a long correspondence with a gentleman of great Mathematical attainments, and who had taken the highest honours at one of our Universities. This correspondence I subsequently published, and when the work came out, I sent copies to some of our leading mathematical *Magnates*. They all kept the book, but not one of them had the courtesy to acknowledge the receipt of it. In the introductory chapter I had spoken of circumstances that occurred at the Aberdeen meeting, under the Presidency of the late—and ever to be lamented—Prince Consort ; and had referred to some of the observations I heard fall from His Royal Highness, in his Presidential address. Under these circumstances, I felt it to be my duty to send his Royal Highness a copy of the work. I was strongly advised not to do so, and told that I might rest assured that no such

work would be permitted to find a place in the library of His Royal Highness. For a time I hesitated : but, ultimately I resolved to transmit a copy to General Grey, the private secretary to His Royal Highness, for presentation ; and was honoured with the following acknowledgment:—

“Mr. Charles Ruland, Librarian to His Royal Highness the Prince Consort, has been commanded to acknowledge the receipt of Mr. James Smith’s work on the Quadrature of the Circle, forwarded through General Grey on the 16th inst.; and at the same time to thank the author for the valuable addition he has made to His Royal Highness’s library.”

Buckingham Palace,
22nd May, 1861.

This acknowledgment was not a mere formality. The Prince was “*learned, wise, sagacious.*”^{*} He read, and discovered the sincerity of the author’s motives ; and appreciating his honest labours in the cause of science, His Royal Highness’s expression of opinion sprang from conviction, and was sincere. I cannot help contrasting this with the treatment I have met with in other quarters ; and, with a deep-felt consciousness of his “*eminent virtues,*”^{*} I shall ever revere the memory of that noble Prince—ALBERT THE GOOD.

I have the honour to be,

Your Grace’s most obedient

And humble Servant,

JAMES SMITH.

^{*} See *Athenæum* : August 3, 1867. Article : The Early Years of His Royal Highness the Prince Consort.

APPENDIX.

(LETTER.)

BARKELEY HOUSE, SEAFORTH,
NEAR LIVERPOOL, 17th June, 1867.

To His Grace the DUKE OF BUCCLEUCH.

My Lord,

May it please your Grace,

I am a very old life Member of *The British Association for the Advancement of Science*, and as such, venture to address your Grace as the President elect. It has been my privilege to discover the solution of the problem:—*What is the true and exact ratio of diameter to circumference in a circle?* But what is of still more importance, this has led me to some most valuable discoveries in astronomical and nautical science. I assume that your Grace, having accepted the distinguished office of President of the British Association for its next meeting, must take a deep interest in such subjects, and I take the liberty of sending your Grace, by this post, a copy of—a French translation of—a pamphlet I distributed at the Oxford meeting of the Association in 1860; and a paper I read before the Literary and Philosophical Society of my native town, in 1864. With regard to the former I may observe, that the translator wrote me asking my permission to publish the translation. This led to an exchange of several letters on the subject, and the translation was published at his own expense. All I know of M. Armand Grange is from his letters, and from them I know him to be an excellent Mathematician. He subsequently went abroad, and I have not heard from him—much to my regret—for the last two years, and now do not know his address.

I may inform your Grace, that the "*guiding stars*" of *The British Association for the Advancement of Science*, will not permit me to read a paper on such subjects in the Physical Section. I am not one of the clique, and have no interest in making a "*mystery*" of science, and "*jealously guarding it.*" Your Grace cannot but be aware of the celebrated letter of Mr. J. R. Hind, the Astronomer, which appeared in the *Times* of the 17th Sept., 1863, in which he places the Earth about four millions of miles nearer the Sun than it has been considered to be for upwards of a century. Then comes Professor Phillips, the President of the British Association, at the Birmingham meeting in 1865, who in his opening address, differs from Mr. J. R. Hind, and maintains that Astronomers have only been wrong by about two millions of miles. And yet these gentlemen are "*guiding stars*" of the British Association, and assume that they have attained the *ne plus ultra* of scientific knowledge, and refuse to listen to anything emanating from a quarter so contemptible in their opinion as the pen of the writer.

The rank and position of your Grace raises you above any such mean and narrow prejudices, and I cannot help thinking the knowledge of these facts will lead your Grace—as the President of the Association for the next year—to enquire into the course the leading members of the Association are pursuing, which, if much longer persisted in, can only have one result. The days of the British Association may, and will be numbered; for no man, or body of men, can for any lengthened period, succeed in stifling truth.

I intend to address a letter to your Grace on the Quadrature and Rectification of the Circle, and hope to send you a copy in time to enable you to give it a perusal before preparing your Presidential Address.

I should be most happy to send your Grace copies of all my published books on these interesting and important topics, should your Grace be pleased to say you will do me the honour to accept them.

I have the honour to subscribe myself,

Your Grace's most obedient

And humble Servant,

JAMES SMITH.

To this letter, I received the following reply :—

DALKEITH HOUSE,
20th June, 1867.

Sir,

I am directed by the Duke of Buccleuch to acknowledge receipt of your letter of the 17th inst.; and to inform you that the packet of papers referred to therein having been addressed to London, has not as yet been forwarded.

His Grace has never taken any part in the arrangements for the meeting of the British Association, and consequently does not know by what rules the Executive are guided in dividing the different subjects into Sections.

I am, Sir,

Your obedient Servant,

JAMES SMITH, ESQ.,

JA. STEUART, Jun.

*Barkeley House, Seaforth,
near Liverpool.*

E R R A T A.

Page 3—For Buccleugh read Buccleuch.

" 11—Eighth line from top, for $23^{\circ} 44'$ read $73^{\circ} 44'$.

" 17—Sixth line from top, for $\frac{4}{3}(\sqrt{19\cdot2})$ read $\frac{5}{4}(\sqrt{19\cdot2})$.

" 30—Ninth line from bottom, for triange read triangle.

" 42—Bottom line, for $\frac{8}{\cdot8081220}$ read $\frac{\cdot8}{\cdot8081220}$.

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